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Stochastic Processes and their Applications 124 (2014) 1335–1367

stochastic processes and their applications

www.elsevier.com/locate/spa

Fractional diffusion limit for a stochastic kinetic equation

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Received 13 March 2013; received in revised form 20 November 2013; accepted 26 November 2013
Available online 4 December 2013

Abstract

We study the stochastic fractional diffusive limit of a kinetic equation involving a small parameter and perturbed by a smooth random term. Generalizing the method of perturbed test functions, under an appropriate scaling for the small parameter, and with the moment method used in the deterministic case, we show the convergence in law to a stochastic fluid limit involving a fractional Laplacian.

Keywords: Kinetic equations; Diffusion limit; Stochastic partial differential equations; Perturbed test functions; Fractional diffusion

1. Introduction

In this paper, we consider the following equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon^{\alpha-1}} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^\alpha} L f^\varepsilon + \frac{1}{\varepsilon^{\frac{\alpha}{2}}} m^\varepsilon f^\varepsilon \quad \text{in } \mathbb{R}_t^+ \times \mathbb{R}_x^d \times \mathbb{R}_v^d, \tag{1.1}$$

with initial condition

$$f^\varepsilon(0) = f_0^\varepsilon \quad \text{in } \mathbb{R}_x^d \times \mathbb{R}_v^d, \tag{1.2}$$

where $0 < \alpha < 2$, L is a linear operator (see (1.3) below) and m^ε a random process depending on $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$ (see Section 2.2). We will study the behaviour in the limit $\varepsilon \to 0$ of its solution f^ε.

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0304-4149/$ - see front matter
http://dx.doi.org/10.1016/j.spa.2013.11.007

The solution $f^\varepsilon(t, x, v)$ to this kinetic equation can be interpreted as a distribution function of particles having position x and degrees of freedom v at time t. The variable v belongs to the velocity space $\mathbb{R}^d$ that we denote by V.

The collision operator L models diffusive and mass-preserving interactions of the particles with the surrounding medium; it is given by

$$Lf = \int_V f \,\mathrm{d}v\, F - f, \tag{1.3}$$

where F is a velocity equilibrium function such that $F \in L^\infty$, $F(-v) = F(v)$, $F > 0$ a.e., $\int_V F(v)\mathrm{d}v = 1$ and which is a power tail distribution

$$F(v) \underset{|v|\to\infty}{\sim} \frac{\kappa_0}{|v|^{d+\alpha}}. \tag{1.4}$$

Note that $F \in \ker(L)$. Power tail distribution functions arise in various contexts, such as astrophysical plasmas or in the study of granular media. For more details on the subject, we refer to [11].

In this paper, we derive a stochastic diffusive limit to the random kinetic model (1.1)–(1.2), using the method of perturbed test functions. This method provides an elegant way of deriving the stochastic diffusive limit from random kinetic systems; it was first introduced by Papanicolaou, Stroock and Varadhan [12]. The book of Fouque, Garnier, Papanicolaou and Solna [7] presents many applications to this method. A generalization in the infinite dimension of the perturbed test function method arose in recent papers of Debussche and Vovelle [6] and de Bouard and Gazeau [8].

For the random kinetic model (1.1)–(1.2), the case $\alpha = 2$ and v replaced by $a(v)$ where a is bounded is derived in the paper of Debussche and Vovelle [6]. Here we study a different scaling parametrized by $0 < \alpha < 2$ and we relax the boundedness hypothesis on a since we study the case $a(v) = v$. Note that, in our case, in order to get a non-trivial limiting equation as ε goes to 0, we exactly must have $a(v)$ unbounded; furthermore, we can easily extend the result to velocities of the form $a(v)$ where a is a C^1-diffeomorphism from V onto V. In the deterministic case, i.e. $m^\varepsilon \equiv 0$, Mellet derived in [10] and [11] with Mouhot and Mischler a diffusion limit to this kinetic equation involving a fractional Laplacian. As a consequence, for the random kinetic problem (1.1)–(1.2), we expect a limiting stochastic equation with a fractional Laplacian.

As in the deterministic case, the fact that the equilibrium F has an appropriate growth when $|v|$ goes to $+\infty$, namely of order $|v|^{-d-\alpha}$, is essential to derive a non-trivial limiting equation when ε goes to 0.

To derive a stochastic diffusive limit to the random kinetic model (1.1)–(1.2), we use a generalization in the infinite dimension of the perturbed test function method. Nevertheless, the fact that the velocities are not bounded gives rise to non-trivial difficulties to control the transport term $v \cdot \nabla_x$. As a result, we also use the moment method applied in [10] in the deterministic case. The moment method consists in working on weak formulations and in introducing new auxiliary problems, namely in the deterministic case

$$\chi^\varepsilon - \varepsilon v \cdot \nabla_x \chi^\varepsilon = \varphi,$$

where φ is some smooth function; thus we introduce in the sequel several additional auxiliary problems to deal with the stochastic part of the kinetic equation. Solving these problems is based on the inversion of the operator $L - \varepsilon A + M$ where M is the infinitesimal generator of the