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The L^2-structures of standard and switching-regime GARCH models

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Abstract

This paper analyzes the probabilistic structure of Markov-switching GARCH(p, q) models, in which the volatility process is driven by a finite state-space Markov chain. We give necessary and sufficient conditions for the existence of moments of any order. We find that the squares and higher order powers of the process have the L^2 structures of ARMA processes, and hence admit ARMA representations. These results are applicable to standard GARCH models and have statistical implications in terms of order identification and parameter estimation.

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1. Introduction

In the statistical analysis of returns on financial assets, one is confronted with the task of modeling time series with a very complicated structure. This structure is characterized by a set of empirical features that are common to many financial data,

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such as stock prices or indices, and exchange rates. These "stylized facts" are lack of serial correlation, heavy-tailed marginal distributions, volatility clustering (large and small observations tend to appear in clusters), and slow decrease of the sample autocorrelations of the squares (or absolute values) of the data. These properties make standard time series models, such as ARMA processes, inappropriate for such series.

The most popular class of volatility models is the autoregressive conditional heteroskedasticity (ARCH) specification introduced by Engle [11] and generalized (GARCH) by Bollerslev [5]. There is a still growing literature on GARCH and related models (see e.g. [2,3,16,21,23] for recent references; see [27] for a comprehensive review on GARCH and stochastic volatility processes). These models have the form $\varepsilon_t = \sqrt{h_t}\eta_t$, where (h_t) is a positive stochastic process, called volatility, and (η_t) is an iid centered sequence, with h_t independent of η_t. Under moment conditions (ε_t) is an uncorrelated sequence. In the standard model, h_t is specified as a linear function of past values of ε_t^2 and h_t. GARCH models gained popularity among practitioners, and in the econometric literature, because they often give a sensible fit to many types of financial data of moderate sample size. However, empirical work has shown that these models have failed to fit the data very well over a long period of time: parameters change, requiring the practitioners to reestimate them.

This observation leads to Markov-switching (MS) GARCH models: GARCH models in which the volatility parameters are assumed to depend on a Markov chain. The MS GARCH(p, q) model we consider is defined by

$$\varepsilon_t = \sqrt{h_t}\eta_t,$$
$$h_t = \omega(\Delta_t) + \sum_{i=1}^{q} \alpha_i(\Delta_t)\varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j(\Delta_t)h_{t-j}, \tag{1}$$

where the parameters are functions of a finite state ergodic Markov chain, denoted by (Δ_t), which is independent of the centered iid sequence (η_t). Precise definitions and assumptions are given in Section 2. MS GARCH models generate series with a much more flexible dependence structure than in standard GARCH specifications. Since the seminal paper by Hamilton [18], the use of Markov-Switching Models has become increasingly popular in econometrics. GARCH models of the type (1), with Markov-switching coefficients, have already been discussed by, for example, Hamilton and Susmel [19], Cai [9], Gray [17], Dueker [10], Francq et al. [12]. A particular case of interest is when (Δ_t) is an independent process (see [30]).

The purpose of this paper is to investigate the L^2 structure of the MS GARCH processes and their even powers. This is a substantial mathematical problem due to the presence of a latent process in the volatility equation. We find that, under existence conditions, the L^2 structure of (ε_t^{2m}), for any nonnegative integer m, is that of a finite ARMA model. This is in particular the case for standard GARCH models. To our knowledge this result was known only for standard GARCH and $m = 1$. We are able to characterize the orders of this ARMA in terms of the MS GARCH