Spatial Statistics 15 (2016) 1–21

Contents lists available at ScienceDirect

Spatial Statistics

journal homepage: www.elsevier.com/locate/spasta

Matérn thinned Cox processes

Ina Trolle Andersen [a,b], Ute Hahn [a,*]

[a] Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark

[b] Stereological Research Laboratory, Department of Clinical Medicine, Aarhus University, Nørrebrogade 44, 8000 Aarhus C, Denmark

ARTICLE INFO

Article history:
Received 4 June 2015
Accepted 15 October 2015
Available online 7 November 2015

Keywords:
Cox process
Dependent thinning
Matérn's hard core process
Pair correlation function
Palm retention probability
Point process

ABSTRACT

A class of spatial point process models that combine short range repulsion with medium range clustering is introduced. The model is motivated by patterns of centres of non-overlapping spherical cells in biological tissue which tend to have a clustering behaviour. Such a combination of clustering and hard core behaviour can be achieved by applying a dependent Matérn thinning to a Cox process. An exact formula for the intensity of a Matérn thinned shot noise Cox process is derived from the Palm distribution. For the more general class of Matérn thinned Cox processes, formulae for the intensity and second-order characteristics are established using the conditional Poisson assumption. These formulae include more complicated integrals for which approximations are suggested to simplify calculations. An example from pathology illustrates the applicability of the models.

1. Introduction

The existing point process literature provides models for a variety of interactions between points, of which the models that allow for simple statistical inference often are those used in applications. Most models yield point patterns with either clustering *or* hard core behaviour; however in practical applications, one may observe both types of interaction on different scales simultaneously. One such example is the pattern formed by centres of cells in cell clusters—the centres cannot come closer than the diameter of the cells, but nevertheless they show clustering on a mid range of spatial distance. This

* Corresponding author.
E-mail addresses: ita@math.au.dk (I.T. Andersen), ute@math.au.dk (U. Hahn).

http://dx.doi.org/10.1016/j.spasta.2015.10.005
2211-6753/

example has motivated the present study. In the literature, such cases are often modelled by Gibbs point processes with an appropriate interaction function, as e.g. in Mattfeldt et al. (2006, 2007). While they have an intuitive physical interpretation through the interaction function, theoretical properties and summary statistics of Gibbs models are accessible only by simulation. Another modelling option is to start with a stationary hard core process and obtain a clustered behaviour by independent thinning with probability according to a random field. Second-order summary statistics of these so-called interrupted point processes (Stoyan, 1979; Lavancier and Møller, in press) can be obtained straightforwardly from the properties of the hard core process and the random field. In particular if the random field can take zero values, such as a Boolean model, this approach yields a clustered appearance. However, the spatial arrangement of points inside the clusters is influenced by "invisible" points outside the clusters that have been thinned from the original homogeneous pattern, which, depending on the application, may seem less natural from a physical point of view.

In the present paper, we introduce and investigate a class of mathematically tractable point process models that combine clustering and hard core property, namely by applying dependent Matérn type II thinning (Matérn, 1960, 1986) to a clustered Cox process (Cox, 1955; Møller and Waagepetersen, 2004). In a nutshell, Matérn's thinning algorithms remove points from an existing pattern that have neighbours that are closer than a given hard core distance h. The thinning condition can be interpreted as the condition that balls with diameter h attached to the points may not overlap. In recent years, generalizations of Matérn hard core models have appeared in the literature. These papers modify the thinning condition, by replacing the non-overlapping balls with more general (random) convex sets (Månsson and Rudemo, 2002; Kiderlen and Hörig, 2013), or by thinning according to more general functions of the distance between points (Teichmann et al., 2013), but they still are confined to thinning a homogeneous Poisson point process, as in Matérn's original work. Very little is known about the point processes resulting from application of Matérn thinning rules to other models. In his Ph.D. thesis (2005), Tscheschel gives a formula for the intensity of Matérn thinned Matérn cluster point processes as a model for microstructure of rubber. With the present paper, we supply a general framework of Palm retention probabilities for calculating both first- and second-order densities of point processes with known Palm distribution in Section 3. For the applications, we focus on Cox point processes as a very flexible class for clustered patterns.

The paper is organized as follows. In Section 2 we give a short theoretical overview of the applied standard point process models, and we recall some of their properties. In Section 3 we derive general expressions for Palm retention probabilities obtained after a Matérn type II thinning procedure. These probabilities are important in the analysis of Matérn thinned Cox processes, defined in Section 4. Sample realizations and theoretical results with respect to Palm retention probabilities, first- and second-order characteristics of the Matérn thinned Cox processes are also presented in Section 4. Simple approximations are suggested in Section 5, for which the quality is supported by simulations of two examples of Matérn thinned Cox processes, the Matérn thinned Matérn cluster process (MCP) and the Matérn thinned Thomas process (TP). The applicability of the proposed class of point processes is illustrated in Section 6 by means of an example from pathology of patterns of megakaryocytes in bone marrow. Finally, a short discussion is found in Section 7 and all the included proofs are found in the Appendix.

2. Preliminaries

This section introduces the notation and basic properties of the point process models considered in this paper. References for detailed description of the theory of point processes include Stoyan et al. (1995), Illian et al. (2008) and Møller and Waagepetersen (2004).

2.1. Spatial point processes

Let N_{lf} denote the set of locally finite subsets of the d-dimensional Euclidean space $\mathbb{R}^d$, equipped with an appropriate σ-algebra $\mathcal{N}_{\text{lf}}$. Then, *a spatial point process* X is a random variable taking values in