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Objective Bayesian analysis of SAR models

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ABSTRACT

The simultaneously autoregressive model (abbreviated as SAR) has been extensively applied for lattice (regional summary) data. A Bayesian approach has been studied by De Oliveira and Song (2008), but they only considered two versions of Jeffreys priors, Jeffreys-rule and independence Jeffreys priors. They recommended the independence Jeffreys prior for a default prior. This prior is known to have the potential problem of posterior impropriety. In this paper, we consider the reference priors including the commonly used reference and "exact" reference priors for the SAR model. We show that common reference priors typically result in improper posterior distributions. Next, two "exact" reference priors are developed and are shown to yield proper posterior distributions. Frequentist properties of inferences based on two "exact" reference and Jeffreys-rule priors are studied by means of simulation. For illustrative purposes, we apply the method to SAT verbal scores across the US.

1. Introduction

One popular model used to investigate the spatial structure of lattice data is the simultaneously autoregressive model (SAR). Whittle (1954) originally developed this model for doubly infinite regular lattices. Since then the SAR model has been used by many other researchers, for example Ord (1975), Doreian (1980, 1982), Pace and Barry (1997), Wall (2004) and Smirnov (2005). The SAR model has been applied for the analysis of data in many areas such as economy, demography and geography.

Maximum likelihood estimates have been the most common method used for fitting the SAR model, but the results from maximum likelihood estimators are asymptotic in nature and it is not known how they behave on small samples. Bayesian analysis has also been studied. For example, Hepple (1995a,b) applied a uniform prior for the model parameters when he considered some aspects of model selection, but he did not verify the propriety of the corresponding posterior

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distribution. LeSage and Paren (2007) considered subjectively proper priors in model selection problems.

De Oliveira and Song (2008) are considered to be the first authors to study objective Bayesian analyses for the SAR models. Within their paper they considered two versions of the Jeffreys prior, the independence Jeffreys prior and the Jeffreys-rule prior. Through a simulation experiment, they found that frequentist properties of inferences based on maximum likelihood (ML) are inferior to those of Bayesian inferences based on any of the three priors: the independence Jeffreys, the Jeffreys-rule and uniform priors. They also found Bayesian inference about the 'spatial parameter' based on the independence Jeffreys prior is slightly better than those based on the Jeffreys-rule prior or uniform prior when the data possess strong spatial association or non-constant mean. Finally, they recommended the independence Jeffreys prior as the default prior. On the other hand, the authors noted that the independence Jeffreys prior does not always yield a proper posterior and pointed out that "posterior impropriety is a potential problem when the independence Jeffreys prior is used", which may pose a problem when applying the method.

Objective Bayesian analysis of spatial data has been extensively studied because the main advantage of the Bayesian technique is that parameter uncertainty is fully accounted for when performing statistical inference and prediction (e.g., Berger et al., 2001; Paulo, 2005; De Oliveira, 2007; Ren et al., 2012). In many cases, only Jeffreys-rule and exact reference priors can yield proper posterior distributions. See Berger et al. (2001), Ren et al. (2012), De Oliveira (2012) and Ren and Sun (in press) for examples. In spite of the success of the Jeffreys-rule prior in one-parameter problems (Welch and Peers, 1963), Berger and Bernardo (1992) gave examples where the Jeffreys-rule prior provided inconsistent estimates in some multiparameter problems. Furthermore, the simulation results for spatial models (e.g., Berger et al., 2001; Ren et al., 2012; Ren and Sun, in press) show that the frequentist performance of the credible sets from reference priors is reasonable while the Jeffreys-rule prior can be seriously inadequate in terms of frequentist performance when the mean function is non-constant. Therefore, it is very important to consider reference priors for the SAR models.

In this paper, we investigate the reference priors for the SAR models including the one derived by De Oliveira and Song (2008), which we present in Part (a) of Proposition 2. Expressing these priors by matrix traces, not by functions of eigenvalues (see Eq. (13)), we get over the difficulties encountered by De Oliveira and Song (2008). Therefore, we obtain the limiting behavior of reference priors as the spatial parameter tends to the boundaries of its range, and the results on the propriety of the resulting posterior distribution.

The paper is organized as follows. Section 2 describes the SAR model. In Section 3, two versions of the Jeffreys prior and their properties De Oliveira and Song (2008) obtained in their paper are summarized. We derive the reference priors and study their properties and the propriety of the corresponding posterior distribution. We compare the three objective priors in Section 4 in terms of ability to produce confidence intervals with good frequentist coverage. The recommended reference prior is illustrated by an example. Conclusions and possible generalizations are given in Section 5. The proof of Proposition 3 is given in the Appendix.

2. The model

Assume that $\{y(\boldsymbol{s}_i) : \boldsymbol{s}_i \in (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)\}$ is a Gaussian random process where $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$ forms a lattice of $\mathcal{D}$. In this paper, we model this process by the simultaneous autoregressive model,

$$y(\boldsymbol{s}_i) = \boldsymbol{x}_i'\boldsymbol{\beta} + \sum_{j=1}^{n} b_{ij}(y(\boldsymbol{s}_j) - \boldsymbol{x}_j'\boldsymbol{\beta}) + e_i, \tag{1}$$

where $E(y(\boldsymbol{s}_i)) = \boldsymbol{x}_i'\boldsymbol{\beta}$, b_{ij} are known or unknown constants and $b_{ii} = 0, i = 1, \ldots, n$, and $\boldsymbol{e} = (e_1, \ldots, e_n)' \sim N(\boldsymbol{0}, \boldsymbol{M})$ are independent. Hence, $\boldsymbol{M}$ is a diagonal matrix. Here $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ip})'$ is a set of p explanatory variables and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ are unknown regression parameters. We call this model *simultaneous* because in general the error terms e_i are correlated with $\{y(\boldsymbol{s}_j) : j \neq i\}$. Thus, the joint distribution of $\boldsymbol{y} = (y(\boldsymbol{s}_1), \ldots, y(\boldsymbol{s}_n))'$ is given by

$$\boldsymbol{y} \sim N(\boldsymbol{X\beta}, (\boldsymbol{I}_n - \boldsymbol{B})^{-1}\boldsymbol{M}(\boldsymbol{I}_n - \boldsymbol{B}')^{-1}), \tag{2}$$