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An introduction to planar random tessellation models

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ABSTRACT

The goal of this paper is to give an overview of random tessellation models. We discuss the classic isotropic Poisson line tessellation in some detail and then move on to more complicated models, including Arak–Clifford–Surgailis polygonal Markov fields and their Gibbs field counterparts, crystal growth models such as the Poisson–Voronoi, Johnson–Mehl and Laguerre random tessellations, and the STIT nesting scheme. An extensive list of references is included as a guide to the literature.

1. Introduction

Random tessellations, also known as random mosaics or stochastic networks, are random partitions of the plane into disjoint regions. Mosaics arise naturally in many contexts. Examples include tilings, crystals, cellular structures, land use maps, galaxies, communication networks, crack patterns, foams, and so on. The use of tessellations has a long history in the geosciences—both as models in their own right and as spatial interpolation tools. See for example Thiessen's classic paper (Thiessen, 1911) on estimating regional rainfall or Harding's work (Harding, 1923) on the estimation of ore reserves. More recent work on related problems includes (Ju et al., 2011; Møller and Skare, 2001). Note that supporting data structures are routinely implemented in GIS systems (Rigaux et al., 2001).

Random tessellations are at the heart of stochastic geometry, the branch of mathematics that concerns itself with modelling and analysing complicated geometrical structures. The aim of this paper is to introduce this fascinating subject to the non-expert and to provide pointers to the literature. For simplicity, all models are described in the plane, but similar models exist in three dimensions.

Formally, a planar tessellation is a collection of mutually disjoint open sets $\{C_1, C_2, \ldots\}$, $C_i \subset \mathbb{R}^2$, such that:

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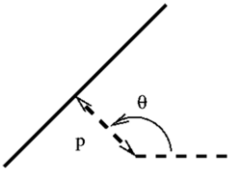

Fig. 1. Parametrisation of lines.

- $C_i \cap C_j = \emptyset$ for $i \neq j$;
- $\cup_i \bar{C}_i = \mathbb{R}^2$;
- for any bounded set $B \subset \mathbb{R}^2$, the set $\{i : C_i \cap B \neq \emptyset\}$ is finite.

Here $\bar{C}$ denotes the topological closure of C. In words, the 'tiles' $\bar{C}_i$ fill the plane, their interiors do not overlap, and only finitely many of them are needed to cover a bounded region. Additional assumptions can be imposed, for example that the sets are non-empty, convex, bounded, or polygons. The C_i are called the cells of the tessellation. Examples are shown in Figs. 2 and 4–6.

Randomness can be generated in a number of ways. For example, one may draw a number of random lines and use them to delineate the boundaries of polygonal cells. Alternatively, a set of points may be generated from which regions or lines are grown until hit by other regions or lines. A pattern of lines may also be used as a skeleton on which to draw non-convex polygonal shapes. Furthermore, operations such as superposition and partitioning can be applied to the cells of a tessellation, either once or as part of an iterative scheme. Most of these mechanisms are briefly discussed in Chapters 10 (sic) of the textbooks (Stoyan et al., 1995; Schneider and Weil, 2008) on stochastic geometry, the lecture notes (Møller, 1994) and the monograph (Okabe et al., 2000) focus on region growing. From a historical perspective the charming booklet (Kendall and Moran, 1963), in summarising classic theory on 'uniformly' distributed random geometrical objects and raising a number of open problems, stimulated research. A partial review of recent developments can be found in Calka (2010).

The plan of this paper is as follows. We first consider Poisson line tessellations in Section 2, then move on to Arak–Clifford–Surgailis polygonal field models in Section 3. Ongoing research on discrete polygonal fields that promise to be useful for image classification and segmentation is touched upon. In Section 4.1, we describe crystal growth models including random Voronoi, Johnson–Mehl and Laguerre tessellations. The new class of stationary iteration stable random tessellations is discussed in Section 4.2. The paper closes with a summary and conclusion.

2. The isotropic Poisson line tessellation

The isotropic Poisson line tessellation is one of the fundamental models in stochastic geometry. In order to describe it, recall that a straight line in the plane can be parametrised by the signed length and orientation of the perpendicular joining the origin with the line; cf. Fig. 1. More specifically,

$$l_{\theta,p} = \{(x, y) \in \mathbb{R}^2 : x \cos\theta + y \sin\theta = p\}$$

for $\theta \in [0, \pi)$, $p \in \mathbb{R}$. The measure μ defined by

$$\mu(E) = \int_E d\theta dp$$

for $E \subseteq [0, \pi) \times \mathbb{R}$ is the unique measure up to a scalar factor that is invariant under rigid motions, that is,

$$\mu(f(E)) = \mu(E)$$

for all f that are compositions of translations and rotations (Poincaré, 1912).

Definition 1. The isotropic Poisson line process L on $\mathbb{R}^2$ with rate $\lambda > 0$ is a Poisson process on $[0, \pi) \times \mathbb{R}$ with intensity measure $\lambda\mu$.