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Multiparametric shell eigenvalue problems

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Abstract

The eigenproblem for thin shells of revolution under uncertainty in material parameters is discussed. Here the focus is on the smallest eigenpairs. Shells of revolution have natural eigenclusters due to symmetries, moreover, the eigenpairs depend on a deterministic parameter, the dimensionless thickness. The stochastic subspace iteration algorithms presented here are capable of resolving the smallest eigenclusters. In the case of random material parameters, it is possible that the eigenmodes cross in the stochastic parameter space. This interesting phenomenon is demonstrated via numerical experiments. Finally, the effect of the chosen material model on the asymptotics in relation to the deterministic parameter is shown to be negligible.

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1. Introduction

In many engineering problems there are uncertainties concerning material models and domains. Stochastic finite element methods (SFEM) have received much attention over the past decade. This, however, has not extended to eigenvalue problems despite their importance in many applications, including the dynamic response of structures which is the focus of this work. In recent paper Sousedik and Elman [1] cover most of the rather limited literature available with the notable exception of work by Andreev and Schwab [2], which is to our knowledge the only mathematically rigorous analysis of the collocation approach to stochastic eigenproblems.

Naturally the focus has been on second order problems where properties such as simple smallest modes can be taken advantage of. For instance, it is well-known that the first mode of the Laplacian does not change its sign. Here the eigenvalue problem, free vibration of thin shells of revolution, is of fourth order where such properties are not guaranteed. Indeed, one feature of such eigenvalue problems that complicates the analysis is the inevitability of repeated or tightly-clustered eigenvalues, which arise naturally with symmetries, and will be central in our numerical experiments. Even in deterministic setting when such eigenvalues are to be approximated in practice, it is futile to try to determine whether computed eigenvalue approximations that are very close to each other are all approximating the

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same (repeated) eigenvalue, or approximating eigenvalues that just happen to be very close to each other. Instead, one should consider such clusters as a collective whole via subspaces.

The main result is that the stochastic subspace iteration is capable of resolving the eigenclusters. Moreover, we demonstrate that eigenvalue crossings occur in the parameter space which leads to the concept of the *effective smallest mode*. For simplicity consider the case where the smallest eigenvalue is of higher multiplicity. Over the parameter space different eigenmodes may be associated with the lowest eigenvalue. In practice this means that over a set of manufactured specimens it is possible to measure or observe different lowest modes due to manufacturing or material imperfections.

Here the shells of revolution are used as the model problem since the eigenmodes have special properties that lend themselves well to our study. The geometry of the shell of revolution is defined by an axial profile function which is rotated about the axis of revolution. In dimensionally reduced setting the midsurface of the shell D has a natural parametrization in the axial and angular directions, $D = [x_0, x_1] \times [0, 2\pi]$ (periodic). If the material properties in the angular direction are constant, the eigenmodes will have integer valued wavenumbers in that direction and using a suitable ansatz the spectrum can be computed over a set of one-dimensional problems. (For an illustration see Fig. 2.) Therefore, introducing uncertainty in the material properties, for instance, Young's modulus, with randomness in the axial direction only is a special case of interest. It should be noted that even though the shell geometry is periodic, it is perfectly feasible to consider material uncertainties that are not continuous as periodic functions. This would correspond to a situation where a cylinder, say, is formed by rolling a cut sheet of material.

Another aspect of the chosen experimental setting is that the numerical *locking* can be controlled. In [3] it is shown that the eigenvalue problem is subject to locking due to angular oscillations. The 1D formulation does not include this form of locking since the integration in the angular direction is exact—assuming that the material parameter varies in the axial direction only. Thus, we can calibrate the 2D discretization to agree with the 1D results. Of course, the rate of convergence cannot be optimal in 2D, but the results can be sufficiently accurate.

Another salient feature of shell problems is the role of dimensionless thickness, which in the context of this paper can be treated as a *deterministic parameter*. It is natural to consider the asymptotic behaviour of the quantities of interest of the stochastic eigenproblems as functions of thickness. Through carefully designed numerical experiments we show that the material imperfections considered here do not affect the known asymptotics for the first eigenpair, and the standard deviation of the smallest eigenvalue decreases linearly as the thickness tends to zero.

The rest of the paper is organized as follows: First in Section 2 the concept of an eigenvalue crossing is illustrated in the context of 2D Laplacian. The shell eigenproblem and its stochastic extension are defined in Sections 3 and 4, respectively. The algorithms necessary (collocation and Galerkin) for the solution of the problems are given in Section 5. The numerical experiments with related analysis of the results are discussed in Section 6. Finally, in Section 7 the conclusions and directions for future research are considered. The shell models used in numerical experiments are outlined in Appendix.

2. On the nature of eigenvalue crossings

The concept of eigenvalue crossings inevitably arises when considering parameter dependent eigenvalue problems. For instance, the second smallest eigenvalue of the Laplace operator on a symmetric domain is of multiplicity two. Thus, as in the example considered in [4] for instance, the second and third eigenvalues of a parameter dependent extension of the problem typically cross within the parameter space.

As an illustrative example we consider the following eigenvalue problem: find $\lambda \in \mathbb{R}$ and $u \in H_0^1(D)$ such that

$$-\nabla \cdot (c\nabla u) = \lambda u,$$

where D is the unit circle in $\mathbb{R}^2$ and $c > 0$. Instead of a constant value for c we consider a random field $c : \Gamma \to L^\infty(D)$ given by

$$c(\xi) = 1 + a_1\xi_1 + a_2\xi_2, \quad \xi = (\xi_1, \xi_2) \in \Gamma := [-1, 1]^2,$$

where $a_1(r, \varphi) = (\cos(\pi r) + 1)/3$ and $a_2(r, \varphi) = \sin(2\varphi)(1 - \cos(\pi r))/3$ in polar coordinates. The eigenpairs (λ, u) now become functions of $\xi \in \Gamma$. We define an ordering of the eigenvalues so that

$$\lambda^{(1)}(\xi) \le \lambda^{(2)}(\xi) \le \dots \quad \forall \xi \in \Gamma \tag{1}$$

and assume the associated eigenfunctions to be normalized in $L^2(D)$ for every $\xi \in \Gamma$.