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Plane thermal transpiration of a rarefied gas in the presence of gravitation

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ABSTRACT

Plane thermal transpiration of a rarefied gas that flows horizontally in the presence of gravitation is studied based on the Boltzmann equation. Assuming that the temperature gradient along the walls is small, the asymptotic analysis for a slow variation in the flow direction is conducted. The semi-analytical solution that is valid for arbitrary values of the mean free path and the gravitational strength is derived, and the problem is reduced to solving the spatially one-dimensional Boltzmann equation. This reduced problem is solved numerically for a hard-sphere molecular gas for small values of gravitational strength, and the behavior of the flow is studied based on the numerical solution. The effect of weak gravitation is no longer negligible when the gas is so rarefied that the mean free path is comparable to the maximum range that the molecules travel along the parabolic path within the channel. This phenomenon has been observed in the plane Poiseuille flow of a highly rarefied gas, and a similar phenomenon also occurs in the plane thermal transpiration considered in the present paper.

1. Introduction

Plane thermal transpiration is the flow of a rarefied gas between parallel plane walls induced by the temperature gradient along the walls. This flow has been studied extensively based on the Boltzmann equation [1–14] because it is a typical rarefaction phenomenon of a gas and it has a variety of applications such as the Knudsen compressor [11,15–19] and the separation of a gas mixture [20]. For small Knudsen numbers (the mean free path divided by the channel width), the asymptotic theory of the Boltzmann equation has been established [11,17]. For the transition regime of the Knudsen number, the direct numerical solution of the linearized Boltzmann equation is available [8,13]. For large Knudsen numbers, a rigorous mathematical analysis of the solution has recently been conducted in Ref. [12]. In Ref. [14], applying the mathematical theorems in Ref. [12], a numerical method for calculating the solution for arbitrary large Knudsen numbers with a rigorous error estimate has been developed. Thanks to this method, the accurate solution of the plane thermal transpiration for a hard-sphere molecular gas over the entire range of the Knudsen number is available. In addition, a comprehensive and accurate experiment for a wide range of the Knudsen number has been reported in Ref. [21].

Recently, the author has studied the horizontal plane Poiseuille flow of a rarefied gas in the presence of gravitation with an arbitrary strength [22]. The effect of gravitation on the flow is studied by means of the asymptotic analysis for a slow variation in the flow direction and the supplemental numerical analysis over a wide range of the Knudsen number and the gravitational strength. Under ordinary room conditions on the earth, the effect of gravitation on the flow is so weak that it is usually negligible in the regime of intermediate Knudsen numbers. However, the effect becomes significant as the Knudsen number increases, as demonstrated in Ref. [22]. The effect of weak gravitation on a highly rarefied gas has been studied intensively in Ref. [23]. As a result, it was clarified that although gravitation is arbitrarily weak, its effect on the flow is no longer negligible when the gas is so rarefied that the mean free path is comparable to the maximum range that the molecules travel along the parabolic path within the channel. The physical explanation of this phenomenon [23] suggests that the plane thermal transpiration considered in the present paper displays a similar behavior. For the understanding of the flow, an intensive study similar to those in Refs. [22,23] is necessary.

In the present paper, we study the horizontal plane thermal transpiration of a rarefied gas in the presence of gravitation on the basis of the Boltzmann equation. Extending the analysis of Ref. [22], the asymptotic analysis for the slow variation in the flow direction is conducted. From the analysis, we obtain the semi-analytical solution that is valid for an arbitrary value of the Knudsen number and the gravitational strength, and the problem is reduced to solving the spatially one-dimensional Boltzmann equation. Then,

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we conduct the numerical analysis of this reduced problem for a hard-sphere molecular gas for small values of gravitational strength. Based on the numerical solution, we discuss the effect of weak gravitation on the plane thermal transpiration of a rarefied gas.

The present paper is organized as follows. In Sec. 2, the problem and the basic equation are stated. In Sec. 3, the asymptotic analysis is conducted. The numerical result is presented and discussed in Sec. 4. Finally, the conclusion is given in Sec. 5.

2. Problem and basic equation

2.1. Problem

Let us consider a rarefied gas between two plane parallel walls placed at rest at $X_2 = 0$ and $X_2 = L$, where X_i is the spatial rectangular coordinate system. The temperature distributions of the two walls are identical and are given as $T_w(X_1)$. The gas is subject to uniform gravitation (0,−g,0). The magnitude of the gravitational acceleration g and that of the mean free path are arbitrary. The conditions at $X_1 = \pm\infty$ are those which maintain the flow time independent, e.g., connected to large reservoirs, and the flow in the region free from the end effects is interested in. We study the time-independent behavior of the gas under the following assumptions: (i) the behavior of the gas is governed by the Boltzmann equation, (ii) the gas molecules undergo diffuse reflection on the walls, and (iii) the temperature gradient of the walls is so small that the quantities vary slowly in the X_1 direction, i.e., $(L/T_w)|\mathrm{d}T_w/\mathrm{d}X_1| \sim \varepsilon(\text{say}) \ll 1$ and the solution varies in this direction in the scale of L/ε.

2.2. Basic equation

In what follows, we use the dimensionless variables $x_i = X_i/L$, $\zeta_i = \xi_i/(2RT_0)^{1/2}$, and $\hat{f} = f/[\rho_0(2RT_0)^{-3/2}]$ for the spatial coordinates X_i, the molecular velocity ξ_i, and the velocity distribution function f. Here T_0 and ρ_0 are arbitrary reference temperature and the reference density, respectively (ρ_0 may be chosen as the average density over the region under consideration). R is the specific gas constant, i.e., the Boltzmann constant divided by the mass m of a molecule. For the convenience of the following analysis, we also use the shrunk coordinate $\chi = \varepsilon x_1$ for x_1.

The time-independent Boltzmann equation for the spatially two-dimensional case is written in the dimensionless form as [11]

$$\varepsilon\zeta_1\frac{\partial\hat{f}}{\partial\chi} + \zeta_2\frac{\partial\hat{f}}{\partial x_2} - \hat{g}\frac{\partial\hat{f}}{\partial\zeta_2} = \frac{2}{\sqrt{\pi}\mathrm{Kn}}\hat{J}(\hat{f},\hat{f}) \tag{1}$$

$$\hat{J}(\hat{f},\hat{f}) = \int\int(\hat{f}'_*\hat{f}' - \hat{f}_*\hat{f})\hat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\mathbf{d}\boldsymbol{\zeta}_* \tag{2}$$

where

$$\hat{f} = f(\boldsymbol{\zeta}),\ \hat{f}_* = f(\boldsymbol{\zeta}_*),\ \hat{f}' = f(\boldsymbol{\zeta}'),\ \hat{f}'_* = f(\boldsymbol{\zeta}'_*)$$
$$\boldsymbol{\zeta}' = \boldsymbol{\zeta} + [\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})]\boldsymbol{\alpha},\quad \boldsymbol{\zeta}'_* = \boldsymbol{\zeta}_* - [\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})]\boldsymbol{\alpha} \tag{3}$$

$\boldsymbol{\alpha}$ is a unit vector, $\mathrm{d}\Omega(\boldsymbol{\alpha})$ is the solid-angle element in the direction of $\boldsymbol{\alpha}$, and $\mathbf{d}\boldsymbol{\zeta}_* = \mathrm{d}\zeta_{1*}\mathrm{d}\zeta_{2*}\mathrm{d}\zeta_{3*}$. $\hat{B} = \hat{B}(|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})|/|\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|, |\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|)$ is a function whose form is determined by the molecular model. The integration in Eq. (2) is carried out over the entire direction of $\boldsymbol{\alpha}$ and the entire space of $\boldsymbol{\zeta}_*$. $\mathrm{Kn} = \ell_0/L$ is the Knudsen number, in which $\ell_0 = 1/(\sqrt{2}\pi d_m^2 m^{-1}\rho_0)$ is the mean free path of the gas in the equilibrium state at rest with the density ρ_0 and d_m is the diameter of a molecule. $\hat{g} = gL/(2RT_0)$ is the dimensionless gravity.

The diffuse reflection boundary condition is written in the dimensionless form as

$$\hat{f} = \left(-2\sqrt{\pi}\int_{\zeta_{2*}<0}\zeta_{2*}\hat{f}_*\mathbf{d}\boldsymbol{\zeta}_*\right)\hat{T}_w^{-2}E\left(\boldsymbol{\zeta}/\hat{T}_w^{1/2}\right)\quad (x_2 = 0,\ \zeta_2>0) \tag{4}$$

$$\hat{f} = \left(2\sqrt{\pi}\int_{\zeta_{2*}>0}\zeta_{2*}\hat{f}_*\mathbf{d}\boldsymbol{\zeta}_*\right)\hat{T}_w^{-2}E\left(\boldsymbol{\zeta}/\hat{T}_w^{1/2}\right)\quad (x_2 = 1,\ \zeta_2<0) \tag{5}$$

where $\hat{T}_w = T_w/T_0$ is the dimensionless temperature distribution along the walls and

$$E(\boldsymbol{\zeta}) = \pi^{-3/2}\exp\left(-\zeta^2\right) \tag{6}$$

The macroscopic variables of the gas, the density ρ, the flow velocity v_i, the temperature T, and the pressure p are defined by the moments of the velocity distribution function. The corresponding dimensionless variables $\hat{\rho} = \rho/\rho_0$, $\hat{v}_i = v_i/(2RT_0)^{1/2}$, $\hat{T} = T/T_0$, and $\hat{p} = p/p_0$, where $p_0 = R\rho_0T_0$, are given as

$$\hat{\rho} = \int\hat{f}\mathbf{d}\boldsymbol{\zeta} \tag{7}$$

$$\hat{v}_i = \frac{1}{\hat{\rho}}\int\zeta_i\hat{f}\mathbf{d}\boldsymbol{\zeta} \tag{8}$$

$$\hat{T} = \frac{2}{3\hat{\rho}}\int(\zeta_i - \hat{v}_i)^2\hat{f}\mathbf{d}\boldsymbol{\zeta} \tag{9}$$

$$\hat{p} = \hat{\rho}\hat{T} \tag{10}$$

The mass flow rate $M(=\int_0^L\rho v_1\mathrm{d}X_2)$ of the gas through a cross section per unit time and per unit width in the X_3 direction is expressed in terms of the dimensionless variables as

$$\frac{M}{2p_0(2RT_0)^{-1/2}L} = \int_0^1\hat{\rho}\hat{v}_1\mathrm{d}x_2 \tag{11}$$

The boundary value problem (1), (4) and (5) is characterized by the two dimensionless parameters

$$\mathrm{Kn} = \frac{\ell_0}{L}\ \text{and}\ \hat{g} = \frac{gL}{2RT_0} \tag{12}$$

and the dimensionless temperature distribution $\hat{T}_w$ along the walls. In the next section, we study this problem for arbitrary values of Kn and $\hat{g}$ and the function $\hat{T}_w$.

3. Asymptotic analysis

In the present section, we seek the solution $\hat{f}$ of the boundary value problem (1), (4) and (5) that varies moderately in the shrunk coordinate χ. The present analysis is a simple application of that in Refs. [11,19,20,24,25,26], and a straightforward extension of that in Ref. [22]. The solution $\hat{f}$ is sought in a power series in ε:

$$\hat{f} = \hat{f}_{(0)} + \hat{f}_{(1)}\varepsilon + \cdots \tag{13}$$