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An asymptotic model for a thin bonded elastic layer coated with an elastic membrane

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ABSTRACT

The deformation problem for a transversely isotropic elastic layer bonded to a rigid substrate and coated with a very thin elastic layer made of another transversely isotropic material is considered. The leading-order asymptotic models (for compressible and incompressible layers) are constructed based on the simplifying assumptions that the generalized plane stress conditions apply to the coating layer, and the flexural stiffness of the coating layer is negligible compared to its tensile stiffness.

1. Introduction

Some natural biological tissues such as articular cartilage possess an inhomogeneous, layered structure with anisotropic material properties. In particular, morphological studies of adult articular cartilage [1,2] show three different zones of preferred collagen fiber bundle orientation. The superficial zone formed by tangentially oriented collagen fibrils provides a thin layer with a high tensile stiffness in the direction parallel to articular surface. It was shown [3] that the high transverse stiffness of the superficial tissue layer (characterized by tangentially oriented collagen fibrils) is important in controlling the deformation response of articular cartilage. Generally speaking, the surface layer in biomaterials usually has different mechanical properties than the underlying bulk material. Due to this circumstance, the mechanical deformation behavior is strongly influenced by the complex interaction of these layers [4].

In the present paper, we consider the deformation problem for a transversely isotropic elastic layer reinforced with a thin elastic membrane ideally attached to one surface, while the other surface is bonded to a rigid substrate. Following Alexandrov et al. [5,6], it is assumed that the reinforcing layer is very thin (with respect to a characteristic size of the applied load) so that its deformation can be treated in the framework of the generalized plane stress state. Moreover, it is assumed that the flexural stiffness of the coating layer is negligible compared to its tensile stiffness. Thus, the reinforcing thin layer is regarded as an elastic membrane.

The rest of the paper is organized as follows. In Section 2, we formulate the three-dimensional boundary conditions for a coated elastic layer. The deformation problem formulation itself is given in Section 3. Asymptotic analysis of the deformation problem is presented in Section 4. The main result of the present paper is presented by the leading-order asymptotic models for

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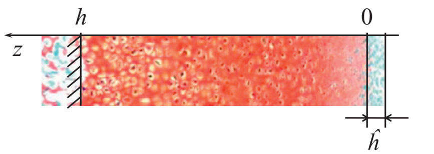

Fig. 1. An elastic coated layer as a model for articular cartilage (the histological image of articular cartilage is taken from the paper [7]). The regions $-\hat{h} \leq z \leq 0$ and $z \geq h$ represent the superficial zone and the subchondral bone, respectively.

the local indentation of the coated elastic layer developed in Section 5 for the cases of compressible and incompressible layer. Finally, Section 7 contains some discussion of the obtained asymptotic models and outlines our conclusion.

2. Boundary conditions for a coated elastic layer

We consider a very thin transversely isotropic elastic coating layer (of uniform thickness $\hat{h}$) ideally attached to an elastic layer (of thickness h) made of another transversely isotropic material (see Fig. 1). Let the five independent elastic constants of the elastic layer and its coating are denoted by $A_{11}, A_{12}, A_{13}, A_{33}, A_{44}$ and $\hat{A}_{11}, \hat{A}_{12}, \hat{A}_{13}, \hat{A}_{33}, \hat{A}_{44}$, respectively.

Under the assumption that the two layers are in perfect contact with one another along their common interface, $z = 0$, the following boundary conditions of continuity (interface conditions of perfect bonding) should be satisfied:

$$\hat{\mathbf{v}}(\mathbf{y}, 0) = \mathbf{v}(\mathbf{y}, 0), \quad \hat{w}(\mathbf{y}, 0) = w(\mathbf{y}, 0), \tag{1}$$

$$\hat{\sigma}_{3j}(\mathbf{y}, 0) = \sigma_{3j}(\mathbf{y}, 0), \quad j = 1, 2, 3. \tag{2}$$

Here, $(\mathbf{v}, \mathbf{w})$ and $(\hat{\mathbf{v}}, \hat{w})$ are the displacement vectors of the elastic layer $z \in (0, h)$ and the elastic coating layer $z \in (-\hat{h}, 0)$, respectively, σ_{ij} and $\hat{\sigma}_{ij}$ are the corresponding components of stress. In what follows, we make use of the Cartesian coordinate system $(\mathbf{y}, z)$, where $\mathbf{y} = (y_1, y_2)$ are the in-plane coordinates.

On the upper surface of the two-layer system, $z = -\hat{h}$, we impose the boundary conditions of normal loading with no tangential tractions

$$\hat{\sigma}_{31}(\mathbf{y}, -\hat{h}) = \sigma_{32}(\mathbf{y}, -\hat{h}) = 0, \quad \hat{\sigma}_{33}(\mathbf{y}, -\hat{h}) = -p(\mathbf{y}), \tag{3}$$

where $p(\mathbf{y})$ is a specified function of external loading.

Following Rahman and Newaz [4], we simplify the deformation analysis of the elastic coating layer based on the following two assumptions: (1) the coating layer is assumed to be very thin, so that the generalized plane stress conditions apply; (2) the flexural stiffness of the coating layer in the z-direction is negligible compared to its tensile stiffness.

In the absence of body forces, the equilibrium equations for an infinitesimal element of the coating layer are

$$\frac{\partial \hat{\sigma}_{j1}}{\partial y_1} + \frac{\partial \hat{\sigma}_{j2}}{\partial y_2} + \frac{\partial \hat{\sigma}_{j3}}{\partial z} = 0, \quad j = 1, 2, 3. \tag{4}$$

The stress-strain relationship for the transversely isotropic elastic coating layer is given by

$$\begin{pmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \\ \hat{\sigma}_{33} \\ \hat{\sigma}_{23} \\ \hat{\sigma}_{13} \\ \hat{\sigma}_{12} \end{pmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & 0 & 0 & 0 \\ \hat{A}_{12} & \hat{A}_{11} & \hat{A}_{13} & 0 & 0 & 0 \\ \hat{A}_{13} & \hat{A}_{13} & \hat{A}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\hat{A}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\hat{A}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\hat{A}_{66} \end{bmatrix} \begin{pmatrix} \hat{\varepsilon}_{11} \\ \hat{\varepsilon}_{22} \\ \hat{\varepsilon}_{33} \\ \hat{\varepsilon}_{23} \\ \hat{\varepsilon}_{13} \\ \hat{\varepsilon}_{12} \end{pmatrix}, \tag{5}$$

where $(\hat{\varepsilon}_{11}, \hat{\varepsilon}_{22}, \ldots, \hat{\varepsilon}_{12})^{\mathsf{T}}$ is the vector of strains in the coating layer, the superscript T denotes the transposition operation, and $2\hat{A}_{66} = \hat{A}_{11} - \hat{A}_{12}$.

Integrating Eq. (4) through the thickness of coating layer and taking into account the interface and boundary conditions (2) and (3), we get

$$\hat{h}\left(\frac{\partial \hat{\bar{\sigma}}_{j1}}{\partial y_1} + \frac{\partial \hat{\bar{\sigma}}_{j2}}{\partial y_2}\right) = -\sigma_{j3}\big|_{z=0}, \quad j = 1, 2, \tag{6}$$

$$\hat{h}\left(\frac{\partial \hat{\bar{\sigma}}_{13}}{\partial y_1} + \frac{\partial \hat{\bar{\sigma}}_{23}}{\partial y_2}\right) = -\sigma_{33}\big|_{z=0} - p. \tag{7}$$

Here, $\hat{\bar{\sigma}}_{ij}$ are the averaged stresses, i.e.,

$$\hat{\bar{\sigma}}_{ij}(\mathbf{y}) = \frac{1}{\hat{h}} \int\limits_{-\hat{h}}^{0} \hat{\sigma}_{ij}(\mathbf{y}, z)\, dz.$$