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Analytical solutions of one-way coupled magnetohydrodynamic free surface flow

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ABSTRACT

We study the flow in a layer of conductive liquid under the influence of surface tension, gravity, and Lorentz forces due to imposed potential differences and transverse magnetic fields, as a function of the Hartmann number, the Bond number, the Reynolds number, the capillary number and the height-to-width ratio A. For aspect ratios $A \ll 1$ and Reynolds numbers $Re \leq A$, lubrication theory is applied to determine the steady state shape of the liquid surface to lowest order. Assuming low Hartmann ($Ha \leq O(1)$), capillary ($Ca \leq O(A^4)$), Bond ($Bo \leq O(A^2)$) numbers and contact angles close to 90°, the flow details below the surface and the free surface elevation for the complete domain are determined analytically using the method of matched asymptotic expansions. The amplitude of the free surface deformation scales linearly with the capillary number and decreases with increasing Bond number, while the shape of the free surface depends on the Bond number and the contact angle condition. The strength of the flow scales linearly with the magnetic field gradient and applied potential difference and vanishes for high aspect ratio layers ($A \to 0$). The analytical model results are confirmed by numerical simulations using a finite volume moving mesh interface tracking method, where the Lorentz force is calculated from the equation for the electric potential. It is shown that the analytical result for the free surface elevation is accurate within 0.4% from the numerical results for $Ha^2 \leq 1$, $Ca \leq A^4$, $Bo \leq A^2$, $Re \leq A$ and $A \leq 0.1$ and within 2% for $A = 0.5$. For $A = 0.1$, the solution remains accurate within 1% of the numerical solution when either Ha^2 is increased to 400, Ca to $200A^4$ or Bo to $100A^2$.

1. Introduction

Magnetohydrodynamic (MHD) free surface flow of a conductive liquid in a spatially non-uniform magnetic field is relevant to various applications in e.g. metallurgy [1–4] and crystal growth processes, such as Czochralski and Bridgman growth [2,5,6]

Wall-bounded single phase MHD flow of a conducting liquid in a magnetic field has been the subject of many theoretical studies, e.g. pipe [7] and duct [8] flow in a uniform magnetic field, convection in a non-uniform magnetic field [9], buoyancy driven Darcy flow in a uniform magnetic field [10–12], and lubrication flow with injected currents [13]. Free surface MHD flow

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has been studied both theoretically and experimentally, e.g. driven by an imposed magnetic field [14], by injected currents [15,16], or due to buoyancy [17].

In the present paper we will study the one-way coupled magnetohydrodynamic (MHD) free surface flow of a conductive fluid in a shallow, two-dimensional cavity subject to a differentially applied electric potential in a spatially non-uniform magnetic field. The Lorentz force will be the only driving force for the flow, while gravity and surface tension act as the restoring forces for the free surface deformation. We will use the analytical methods of lubrication theory and matched asymptotic expansions to determine the free surface elevation and flow inside the cavity. This combination of methods has been used previously to study flows in shallow cavities, for example in buoyancy driven single phase flow [18], and later for free surface flow driven by a Marangoni force [19]. It has also been used for single phase MHD flow [20] and free surface MHD flow in a uniform magnetic field [21]. The combination of a free surface and a non-uniform magnetic field distinguishes our work from previous studies.

Our analytical solutions will subsequently be compared with numerical results from a finite volume based free surface, one-way coupled Navier–Stokes MHD flow solver. The free surface is modelled using a moving mesh interface tracking (MMIT) method and the electric potential in the one-way coupled MHD problem is calculated from a Poisson equation.

The goal of this paper is to (i) find asymptotic analytical solutions for the flow in a conductive layer of fluid influenced by Lorentz, gravity and surface tension forces, (ii) validate a free surface MHD flow solver in OpenFoam [22] against the obtained asymptotic analytical solutions and (iii) use the validated numerical solver to explore the parameter space, in terms of Hartmann number, capillary number, Bond number, Reynolds number and aspect ratio, for which the analytical solution is accurate. With the obtained knowledge about its accuracy and limitations, the presented asymptotic analytical solutions may subsequently serve as a benchmark for the validation of other numerical solvers for combined free surface and MHD flows.

This paper is outlined as follows. The mathematical framework is presented in Section 2, this includes the derivation of the flow in the core and the free surface elevation. In Section 3, the turning flow near the side walls is derived analytically. Section 4 demonstrates the applicability of the mathematical model for increasing magnetic field strengths. Section 5 introduces the numerical method used in this paper and Section 6 shows the verification of the mathematical solutions by means of the numerical model.

2. Mathematical model

We consider a two-dimensional, finite-size liquid layer of width l and initial height d, as depicted in Fig. 1. The liquid has an electrical conductivity σ, density ρ and kinematic viscosity ν. The fluid above the liquid layer is assumed to have negligible electrical conductivity, density and viscosity. The surface tension between the two phases is denoted by γ and the downward directed gravitational force by g.

The left wall of the system is kept at a fixed electrical potential $-\frac{1}{2}\Delta\phi$, the right wall at $\frac{1}{2}\Delta\phi$ and the bottom wall is electrically insulated. A magnetic field $\mathbf{b}'/b_0 = -(\alpha z'/l)\hat{\mathbf{x}} - (1+\alpha x'/l)\hat{\mathbf{z}}$ is imposed, which in the plane of interest, the $z = 0$ plane, gives a linearly increasing magnetic field $\mathbf{b}' = -b_0(1+\alpha x'/l)\hat{\mathbf{z}}$. The Lorentz force associated with this magnetic field has a zero z-component in the $z = 0$ plane.

In equilibrium, a net current flows from the right to the left wall. This imposed current, which is closed externally, will later be shown to be an order of magnitude stronger than the induced current, and, which due to its interaction with the magnetic field, leads to a Lorentz force $\mathbf{f}' = \mathbf{j}' \times \mathbf{b}'$. This causes a net downward force on the conducting liquid, which is stronger at the right side than at the left side. This will initiate a circulating flow inside the fluid that via pressure build-up deforms the interface. Viscous, gravitational and surface tension forces act to oppose the Lorentz force.

2.1. Conservation equations

The problem will be studied from the conservation equations for mass, momentum and current, which, for an incompressible, non-Newtonian fluid, read

$$\nabla' \cdot \mathbf{u}' = 0, \quad (1)$$

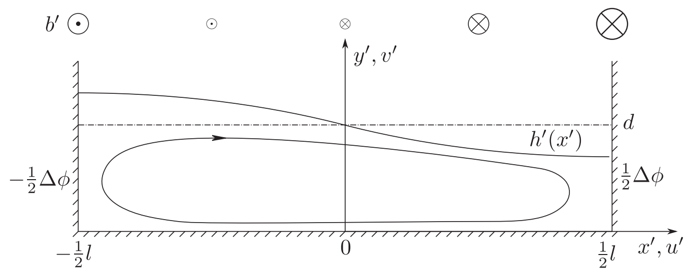

Fig. 1. Schematic representation of the liquid layer, with a potential difference $\Delta\phi$ across the domain and an insulated bottom. The dash-dotted line is the undisturbed interface and the solid line $h(x)$ is the equilibrium surface position after application of a magnetic field in the direction perpendicular to the xy-plane, where the relative field strength is indicated in the top.