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Memristive *jounce* (Newtonian) circuits

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ABSTRACT

Memristive circuits with mixed-mode oscillations together with the regular and canonical Chua's circuits and their nonlinear mathematical models of order four are analyzed in this paper. The circuits are linked to Newton's second law $u'' - F(t,u,u')/m = 0$ for all dependent variables with the nonlinear *force* functions F containing memory terms. The nonlinear memristive element in each circuit is described by $y = g(w)x,\ w' = x,$ where x, y and w are the current, voltage and flux, respectively, in some of the circuits, and the voltage, current and charge, respectively, in others. Because of the link of the circuits to Newton's second law it is possible to interpret the fourth derivatives of the dependent variables in the circuits as the *jounce* variables – the fourth derivative of the position variable in dynamical mechanical systems.

1. Introduction

A renewed interest in memristors and memristive circuits has been developing after the HP lab's announcement of a successful construction of a memristive element with a charge dependent memristance [1]. Several interesting circuits with memristive elements have been proposed and their unusual dynamical properties have been analyzed, among others, in [2–8]. The well-known regular and canonical Chua's circuits have been modified to include memristive elements to yield oscillatory responses (including chaos) [9–12]. Other memristive circuits yield periodic mixed-mode oscillations (MMOs) [13,14].

MMOs are sequences of both small- and large-amplitude oscillations (or SAOs and LAOs) occurring in chemical, electrical, biological, mechanical and astrophysical systems, including dysrhythmias of human hearts and epileptic synchronizations of neurons in human brains [15–21]. A typical system with MMOs is of third order and has a cubic-type polynomial nonlinearity. Such systems are sometimes called the *jerk* systems, since, in physics and mechanics the third derivative of position is called the *jerk*. Various aspects of *jerk* systems, their properties and circuit realizations have been studied in [22–26]. When a nonlinear element with current-voltage characteristic in such systems and circuits is replaced by a memristive element described by the output-input relationship $y = g(w)x$ and internal variable w such that $w' = x$, then the order of the system increases by one, so a typical third-order system becomes a fourth-order one, as, for example, in the cases analyzed in [9,13].

In this paper, we further analyze the fourth-order memristive circuits with MMOs proposed recently in [13,14] and study their properties in the context of Newton's second law. In particular, we show that each dependent variable in those circuits can be described by Newton's second law $u'' = F(t,u,u')/m$ for constant m and a nonlinear and non-autonomous *force* function F containing memory terms. Each such equation when differentiated twice with respect to the time variable t results in a fourth-order ODE with the derivative $u''''(t)$, which can be interpreted as a *jounce* variable, that is the fourth derivative of the *position*

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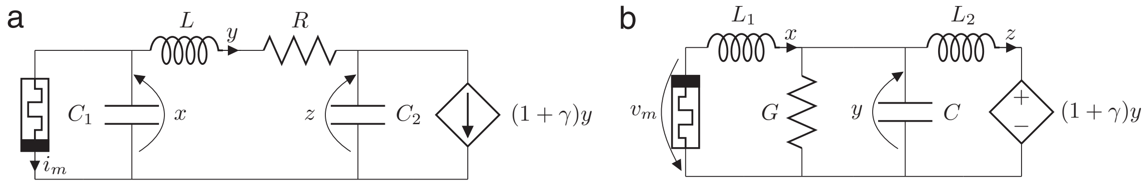

Fig. 1. Two dual memristive circuits described by (1) with $x(t) = \eta\bar{x}(t)$.

variable $u(t)$. Any such ODE can be called a *jounce* (Newtonian) ODE, for the same reason that the third-order ODEs and the corresponding circuits are called the *jerk* ODEs and circuits. This link of memristive MMO circuits with Newton's second law allows to link electrical variables to their mechanical equivalents and possibly to further uncover and interpret the hidden links between certain nonlinear phenomena in memristive circuits with those specific only for mechanical systems.

An important feature of two MMO *jounce* circuits considered in this paper is the singularity perturbed character of the circuits, which results in periodic sequences of both SAOs and LAOs in various patterns. Such a singularly perturbed nature of the circuits is due to a *small* parameter $0 < \epsilon \ll 1$ (either a capacitance or inductance) present in the circuits. Also, the well-known Chua's circuits with the Chua's diodes replaced by memristors can be considered as *jounce* (Newtonian) circuits, as analyzed in this paper.

The paper is organized as follows. In Section 2 we briefly describe the two memristve MMO circuits. Section 3 contains analysis of mathematical properties of the *jounce* memristive circuits with MMOs. The detailed derivations of Newton's second laws for each of the four dependent variables of the circuits is provided in that section. Section 4 is focused on derivation of Newton's second laws for the regular and canonical Chua's circuits. Section 5 contains concluding remarks.

2. The MMO memristive circuits

Very little is known about the possibility of transforming a system of four nonlinear ODEs into scalar ODEs. Certainly, much more is known about the same problem for systems of three equations, see [22,23] for a few answers to such a question.

Consider the two dual memristive circuits introduced recently in [13] and shown in Fig. 1. Both circuits are described by the following system of four ODEs in which only the first equation is nonlinear (due to a nonlinear mem-element)

$$\begin{aligned} \epsilon\bar{x}' &= s_c[-y/\eta - g(w)\bar{x}] \\ y' &= s_c\alpha(\eta\bar{x} - Ky - z) \\ z' &= -s_c\beta y \\ w' &= s_c\eta\bar{x} \end{aligned} \tag{1}$$

where the prime $'$ denotes the time derivative, $0 < C_1 \equiv \epsilon \ll 1$, $\alpha = 1/L$, $K = R$, $\beta = \gamma/C_2$ for the circuit in Fig. 1(a) and $0 < L_1 \equiv \epsilon \ll 1$, $\alpha = 1/C$, $K = G$, $\beta = \gamma/L_2$ for the circuit in Fig. 1(b). The current-controlled current source and voltage-controlled voltage source in the circuits are described through the expression $(1+\gamma)y$ with $\gamma > 0$. The scaling factor $\eta > 1$ was chosen to reduce the variable x and its derivative (important in circuit simulation in SPICE [27]), since $\bar{x} = x/\eta$, with x being the memductor's voltage in the circuit in Fig. 1(a) and memristor's current in the circuit in Fig. 1(b). The $s_c > 0$ is a time-scaling coefficient. In some examples in this paper we also consider (1) with the second equation replaced by $y' = s_c\alpha(\eta\bar{x} - Ky - z \pm a_s)$, that is a small biasing constant source a_s of order ϵ is used. The a_s is a voltage source added to the *RL* branch in Fig. 1(a) and a parallel current source added to the branches of *GC* in Fig. 1(b). When $a_s \neq 0$ one can use zero-initial conditions to obtain MMOs. Otherwise, with $a_s = 0$, non-zero initial conditions should be used (for example $[x(0), y(0), z(0), w(0)] = [2.22, 1.105, -0.00628, 0.375]$ from [13,14]). The current through the memductor in Fig. 1(a) equals $g(w)x$, where w and x are the flux and voltage, respectively. For the dual circuit in Fig. 1(b) the voltage equals $g(w)x$ with w and x being the memristor's charge and current. A general integrable function $g(w)$ is considered in the mathematical analysis of (1) in Section 3 and also in Chua's circuits in Section 4.

The analysis of (1) and derivation of Newton's second law for variables x, y, z and w are presented in the next section. The same is performed for the Chua circuits in Section 4.

3. Newtonian properties and the *jounce* ODEs

In this section, we analyze the fourth-order MMO memristive circuits in Fig. 1 and their mathematical model (1), for a general smooth (C^4 suffices) function $g(w)$. Some preliminary properties of the circuits (MMOs, pinched hysteresis loops, bifurcations) have been reported in [13,14]. Now, we extend those properties in the context of Newton's second law. In particular, we show that each dependent variable x, y, z and w can be described by Newton's law in the general form $u'' = F(t, u, u')/m$ with constant m and a nonlinear/non-autonomous *force* function F containing memory terms. We also formulate the *jounce* (or *snap*) (instantaneous rate of change of *jerk*) equations for each variable, which should be valuable in the study of Newtonian jouncy dynamics as well as possible chaotic motion.

Using the transformation $x = \eta\bar{x}$ and the second equation in (1) with a_s (as described in Section 2), we have the following.