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A new treatment based on hybrid functions to the solution of telegraph equations of fractional order

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ABSTRACT

In this paper, a new operational method based on hybrid functions of Legendre polynomials and Block-Pulse-Functions will be presented. The operational matrix of fractional integration is derived and used to take an acceptable approximate for the solution of a telegraph equation of fractional order. An error estimation will be presented to give an image of the goodness of the solution. Some numerical examples demonstrate the efficiency of the proposed method.

1. Introduction

A class of hyperbolic partial differential equations which describes vibrations within objects and how waves are propagated, is called Telegraph equation [1]. Equations of the form of telegraph equations arise in the study of propagation of electrical signals in a cable of transmission line and wave phenomena. Interaction between convection and diffusion or reciprocal action of reaction and diffusion describes a number of nonlinear phenomena in physical, chemical and biological process [2–5]. In fact, the telegraph equation is more suitable than ordinary diffusion equation in modeling reaction diffusion for such branches of sciences. For example, biologists encounter these equations in the study of pulsatileblood flow in arteries and in one-dimensional random motion of bugs along a hedge [6]. Also the propagation of acoustic waves in Darcy-type porous media [7], and parallel flows of viscous Maxwell fluids [8] are just some of the phenomena governed by the telegraph equations [9–11]. The telegraph equation has also been employed in other areas. For example, it is used as a replacement for the diffusion equation to model transport of charged particles [12,13], high frequency transmission lines [14,15], solar cosmic rays [16], chemical diffusion, anomalous diffusion [17,18], hydrology and population dynamics [19]. It is also employed in the theory of hyperbolic heat transfer [20,21].

Finite difference methods are known as the first techniques for solving partial differential equations [22,23]. Even though these methods are very effective for solving various kinds of partial differential equations. Many researchers have used various numerical and analytical methods to solve the telegraph equation [24,25]. Unconditionally stable and parallel difference scheme, Chebyshev Tau method, Chebyshev method, Legendre multi-wavelet Galerkin method, meshless local weak– strong methods, homotopy analysis and Adomian decomposition are such methods [12,15,26–36].

The fractional calculus is one of the most accurate tools to refine the description of natural phenomena [37]. Fractional differential equations have attracted in the recent years a considerable interest due to their frequent appearance in various fields

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and their more accurate models of systems under consideration provided by fractional derivatives [38,39]. Fractional telegraph equations has been recently considered by many authors. Cascaval et al. [40] discussed the time-fractional telegraph equations, dealing with well posedness and presenting a study of their asymptotic behavior by using the Riemann– Liouville approach. Orsingher and Beghin [41] discussed the time-fractional telegraph equation and telegraph processes with Brownian time, showing that some processes are governed by time-fractional telegraph equations. Chen et al. [42] examined and derived a solution of the time-fractional telegraph equation with three kinds of nonhomogeneous boundary conditions, by the method of separation of variables. Fractional telegraph equations from the analytic point of view have been studied by many authors (see Saxena et al. [43] for equations with n time derivatives).

The aim of this paper is to introduce a new method for approximating the solution of a time-fractional telegraph equation in the following form:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)+\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}u(x,t)+u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t)=f(x,t), \tag{1.1}$$

with the initial and boundary conditions:

$$u(x,0)=l_1(x),\quad u(0,t)=g_1(t),\quad u(x,1)=l_2(x),\quad u(1,t)=g_2(t). \tag{1.2}$$

The right-hand-side function $f(x, t)$ is given and $1 < \alpha \le 2$ is a real number.

In this paper, first hybrid functions of Legendre polynomials and Block-Pulse-functions are introduced and a new operational matrix of fractional integration is constructed. Then a numerical method to solve Eq. (1.1) will be developed. Also some error analysis will be presented. Finally, some numerical examples are presented to confirm the applicability of the method.

2. Real-world application of telegraph equations

In the following we mention some real-world applications of telegraph equations.

(1) Consider the following fractional telegraph equation [41]:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}+2\lambda\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=c^2\frac{\partial^2u(x,t)}{\partial x^2},\quad 0<\alpha\le 1, \tag{2.1}$$

with the conditions:

$$\begin{cases} 0<\alpha\le\dfrac{1}{2}: & u(x,0)=\delta(x),\\ \frac{1}{2}<\alpha\le 1: & u_t(x,0)=0.\end{cases}$$

Here, we consider the case $\alpha = \frac{1}{2}$ for which it is possible to obtain the solution. The probability density of the solution of (2.1) coincides with the distribution of the telegraph process $T = T(t), t > 0$ with a Brownian time, that is:

$$W(t)=T(|B(t)|).$$

This means that the fundamental solution to the fractional Eq. (2.1) for $\alpha = \frac{1}{2}$ can be interpreted as the distribution of a particle moving back and forth on the real line with velocities $\pm c$ (switching of Poisson-paced time) for a random time interval of length $|B(t)|$. Clearly, B and T are independent of each other. Eq. (2.1) for $\alpha = \frac{1}{2}$ is a heat equation with a damping term depending on all values of u in $[0, t]$ and assigning an overwhelming weight to those close to t. The damping effect of $\partial^{\frac{1}{2}}u/\partial t^{\frac{1}{2}}$ reverberates on the probability density of the solution, where the governing term (solution to the heat equation) is weighted by the telegraph distribution (representing the impact of the fractional derivative).

(2) The transmission line equation can be represented as:

$$\frac{\partial V(z,t)}{\partial z}=-RI(z,t)-L\frac{\partial I(z,t)}{\partial t}, \tag{2.2}$$

$$\frac{\partial I(z,t)}{\partial z}=-GV(z,t)-C\frac{\partial V(z,t)}{\partial t}, \tag{2.3}$$

where R, G, C, and L are the per-unit-length parameters of the transmission line [5]. The $V(z, t)$ and $I(z, t)$ are the line voltages (with respect to the reference line) and line currents, respectively. Fig. (1) shows a segment of a typical transmission line. Differentiating (2.2) and (2.3) with respect to z, we get:

$$\frac{\partial^2V(z,t)}{\partial z^2}=RGV+(RC+LG)\frac{\partial V(z,t)}{\partial t}+LC\frac{\partial^2V(z,t)}{\partial t^2}, \tag{2.4}$$

$$\frac{\partial^2I(z,t)}{\partial z^2}=RGI+(RC+LG)\frac{\partial I(z,t)}{\partial t}+LC\frac{\partial^2I(z,t)}{\partial t^2}, \tag{2.5}$$

which are two telegraph equations for $\alpha = 2$.

In [14], the authors have solved the following example.

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