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# Well-balanced central schemes for systems of shallow water equations with wet and dry states

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ABSTRACT

In this paper we propose a new well-balanced unstaggered central finite volume scheme for the shallow water equations on variable bottom topographies, with wet and dry states. Based on a special piecewise linear reconstruction of the cell-centered numerical solution and a careful discretization of the system of partial differential equations, the proposed numerical scheme ensures both a well-balanced discretization and the positivity requirement of the water height component. More precisely, the well-balanced requirement is fulfilled by following the surface gradient method, while the positivity requirement of the computed water height component is ensured by following a new technique specially designed for the unstaggered central schemes. The developed scheme is then validated and classical shallow water equation problems on variable bottom topographies with wet and dry states are successfully solved. The reported numerical results confirm the potential and efficiency of the proposed method.



## 1. Introduction

Introduced by Nessyahu and Tadmor [1] in 1990, central schemes were meant to be simple and efficient tools for the numerical solution of systems of hyperbolic conservation laws. Based on the staggered Lax–Friedrichs scheme, the Nessyahu and Tadmor (NT) central scheme evolves a piecewise linear numerical solution on two staggered grids and avoids the time consuming process of solving Riemann problems arising at the cell interfaces. Furthermore, second-order quadrature rules along with a gradients limiting process result in a second-order accurate and oscillations-free finite volume method for general hyperbolic systems of conservation laws. Multidimensional extensions of the NT scheme on Cartesian grids [2–5] and unstructured grids [6–9] were later developed and successfully applied to solve hyperbolic arising in aerodynamics and magnetohydrodynamics [5,10], as well as balance law problems arising in hydrodynamics [11–13].

One main disadvantage of using central schemes remains in the fact that a dual staggered grid is required to evolve the numerical solution, and this leads to synchronization issues whenever physical constraints are to be numerically forced, or if steady states are to be handled in the case of balance laws. To overcome this problem, Jiang et al. [14] presented a first adaptation of the NT scheme that evolves the numerical solution on a single grid and another adaptation (known as unstaggered central scheme UCS) followed in [13]. Based on a careful projection of the numerical solution obtained on the staggered cells, back onto the original cells, the UCS method was successfully used to solve steady state problems arising in hydrodynamics, and ideal/shallow water magnetohydrodynamics [15,16]. In this paper we develop a new well-balanced second-order accurate unstaggered central scheme for the numerical solution of shallow water equation (SWE) problems with wet and dry states. The SWE system describes

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the motion of a free surface incompressible fluid under gravity, over a variable bottom topography. The SWE system is widely accepted as a mathematical model for the hydrodynamics of coastal oceans, simulation of flows in rivers, and can also be used to simulate tsunami and inundation waves, dam breaches, and others. Ignoring the friction effects, the SWE system can be written as a system of hyperbolic conservation laws with a source term describing the effects of the varying water bed or bathymetry; this source term vanishes in the case of a flat bathymetry. Recently, the research on numerical methods for the shallow water equations became popular for two main reasons or challenges: The SWE system features equilibrium states/stationary solutions in which the nonzero flux divergence is exactly balanced by the source term. In their native form, usually most numerical schemes fail to generate equilibrium state solutions and they generate nonphysical waves, oscillations, and instabilities because the flux divergence and the source term lack well-balancing in their discretization. Well-balanced schemes were developed in [12,17–26] in a way to satisfy the steady state requirement such as the lake at rest problem. Another important feature in the simulation of the SWE problems is the appearance of wet and dry areas that are due to the initial conditions or to the flow of water in the computational domain. Here again most numerical schemes fail to handle the interaction between wet and dry zones and generate negative water heights and other instabilities. The development of well-balanced schemes with wetting and drying capabilities formed another challenge among the numerical community, and several numerical schemes were recently developed [27–30] to fulfill both the lake at rest and the wetting and drying constraints.

In this paper we develop, analyze and implement a new unstaggered, well-balanced, non-oscillatory, and second-order accurate central scheme for the one-dimensional system of shallow water equations on irregular bathymetry with wet and dry zones. The lake at rest constraint will be exactly satisfied at the discrete level by following the surface gradient method developed in [12,26] which discretizes the water height according to the discretizations of the water level and the bottom topography functions. On the other hand wet and dry zones will be carefully treated by introducing a new technique that corrects the slopes of the linearized water height function over the control cells in the forward and backward projections steps in a way to ensure both water conservation and non-negative water height values. We show that the resulting scheme is a well-balanced scheme that exactly maintains the steady state requirement at the discrete level when lake at rest problems are considered and also allows proper wave run-ups and withdraws on coastal slopes and shorelines.

The rest of the paper is organized as follows: Section 2 is dedicated to reviewing the shallow water equations, their properties, and their equilibrium states and constraints. In Section 3 we develop the well-balanced central scheme for the shallow water equations with wet and dry states; we describe the well-balancing technique that ensures the lake at rest constraint and the wet/dry treatment that allows a proper propagation of water waves on shores and islands. Section 4 is devoted to the validation and application of the developed scheme; we perform several numerical experiments from the recent literature, and we confirm the robustness and potential of the proposed scheme. We end the paper with some concluding remarks and perspectives for future work in Section 5.

## 2. Shallow water equations

Shallow water equations are commonly used to mathematically model rapidly varying free surface flows such as dam breaches, floods and inundation waves, tidal waves in oceans and lakes and many others. The system of shallow water equations is derived from the conservation principles of the mass and momentum, and under the main assumption of hydrostatic pressure distribution, the SWE system is a time dependent two-dimensional system of hyperbolic balance laws. The conservative one-dimensional version of the SWE system reduces to

$$\begin{cases} \partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = S(\mathbf{u}, x), \quad t > 0, x \in \Omega \subset \mathbb{R}, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x), \end{cases} \tag{1}$$

with $x$ and $t$ are the spatial and temporal independent variables, respectively. The computational domain $\Omega$ is an interval of the real axis. Furthermore, the unknown vector solution $\mathbf{u}(x, t)$, the flux function $f(\mathbf{u})$, and the source term $S(\mathbf{u}, x)$ are given as follows:

$$\mathbf{u}(x,t) = \begin{pmatrix} h \\ hv \end{pmatrix}, \; f(\mathbf{u}) = \begin{pmatrix} hv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix}, \text{ and } S(\mathbf{u}, x) = \begin{pmatrix} 0 \\ -gh\frac{dz}{dx} \end{pmatrix}. \tag{2}$$

Here $h(x, t)$ denotes the water height, $v(x, t)$ is the velocity in the $x$-direction, $g$ is the gravitational constant, and $z(x)$ denotes the bottom topography function (Fig. 1). When the waterbed is flat, i.e. $z(x) =$ constant, the right hand side of system (1) vanishes and the resulting system reduces to a homogeneous hyperbolic system with real eigenvalues $\lambda_1 = v - \sqrt{gh}$ and $\lambda_2 = v + \sqrt{gh}$ and linearly independent eigenvectors.

The SWE system features equilibrium solutions that are solutions to the system

$$\begin{cases} \partial_x(hv) = 0, \\ \partial_x(hv^2 + gh^2/2) = -dh\frac{dz}{dx}. \end{cases} \tag{3}$$

One particular equilibrium state that we would like to satisfy in this work is the lake at rest equilibrium state defined by

$$\begin{cases} v = 0, \\ h + z = \text{constant}. \end{cases} \tag{4}$$

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