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Decomposition methods for coupled 3D equations of applied mathematics and continuum mechanics: Partial survey, classification, new results, and generalizations

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ABSTRACT

The present paper provides a systematic treatment of various decomposition methods for linear (and some model nonlinear) systems of coupled three-dimensional partial differential equations of a fairly general form. Special cases of the systems considered are commonly used in applied mathematics, continuum mechanics, and physics. The methods in question are based on the decomposition (splitting) of a system of equations into a few simpler subsystems or independent equations. We show that in the absence of mass forces the solution of the system of four three-dimensional stationary and nonstationary equations considered can be expressed via solutions of three independent equations (two of which having a similar form) in a number of ways. The notion of decomposition order is introduced. Various decomposition methods of the first, second, and higher orders are described. To illustrate the capabilities of the methods, more than fifteen distinct systems of coupled 3D equations are discussed which describe viscoelastic incompressible fluids, compressible barotropic fluids, thermoelasticity, thermoviscoelasticity, electromagnetic fields, etc. The results obtained may be useful when constructing exact and numerical solutions of linear problems in continuum mechanics and physics as well as when testing numerical and approximate methods for linear and some nonlinear problems.

1. Introduction: Some coupled equations of continuum mechanics and physics

1.1. Preliminary remarks

Linear systems of coupled equations arise in various areas of continuum mechanics and physics, namely, in elasticity, thermoelasticity, poroelasticity, the theory of viscous and viscoelastic incompressible fluids and viscous compressible barotropic fluids, electrodynamics, etc.

It is always deemed to be substantial progress if one manages to decompose a linear or nonlinear system of coupled equations into a few simpler subsystems or even reduce it to several independent equations. In this paper, a representation of solutions to a system of coupled equations via solutions of independent (uncoupled) equations is called a *complete decomposition* of the

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original system, and a representation of solutions to a system of coupled equations via solutions of a few simpler equations (only some of which are independent) is called an *incomplete* (or *partial*) *decomposition*.

Decomposition simplifies the qualitative study and interpretation of the most important physical properties of coupled 3D equations dramatically and has come to play a pervasive role in understanding their wave and dissipative properties. Furthermore, decomposition often permits finding exact closed-form solutions of the corresponding boundary and initial-boundary value problems and considerably simplifies the application of numerical methods, allowing one to use appropriate software for the simpler independent equations or subsystems.

The present paper stems from a very classical part of continuum mechanics. Various transformation methods simplifying (in particular, decoupling) the linear equations of elasticity and hydrodynamics have undergone extensive development since the middle nineteenth century. A number of such transformations can be found in the classical monographs dealing with theoretical elasticity [1–5], hydrodynamics [6–8], general field theories of mechanics [9,10], etc., but the reasoning there is usually based on intuitive specific properties of particular systems. Furthermore, attention is mainly focused on the equations of linear elasticity. Decomposition methods for the equations of thermoelasticity and poroelasticity have been studied much less comprehensively [11–15]. There have been only a few extensions of these concepts to systems of general form (e.g., see [16,17]), although pioneering results were obtained in as early as 1900 by Duhem [18]. Relevant references to the original and later results in this area will be given in the text below.

The present paper proposes a systematic approach to the decomposition of systems of equations for various classes of three-dimensional linear (and model nonlinear) equations and provides specific examples for systems of equations arising in elasticity, thermoelasticity, thermoviscoelasticity, and the mechanics of a viscous or viscoelastic incompressible or compressible barotropic fluids.

Remark 1. It is important that the methods discussed in the paper are exact and do not use any approximations or additional a priori assumptions about the structure of the solutions sought (unlike numerical and approximate analytical methods). The methods concerned may be useful when constructing exact and numerical solutions of various linear problems in continuum mechanics and physics. The results presented in this paper may serve as a basis for constructing numerous 3D test problems to evaluate the accuracy of numerical and approximate analytical methods.

In what follows, we write out all equations in Cartesian coordinates x, y, and z; the equations may also involve time t; all functions occurring in the original and resulting equations are assumed to be sufficiently smooth with respect to x, y, z, and t. We also use the vector notation $\mathbf{x} = (x, y, z)$.

We begin with a brief informative description of special cases of equation system commonly used in continuum mechanics and physics. Then we proceed to a general system of coupled equations considered in the main body of the paper.

1.2. Equations of linear elastodynamics

Consider the linear equations governing the motion of an elastic isotropic homogeneous medium [19,20]:

$$\rho\mathbf{u}_{tt} = \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\,\mathrm{div}\,\mathbf{u} + \rho\mathbf{f}, \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement field, t is time, ρ is the medium density, λ and μ are the Lamé elastic moduli (μ is the shear modulus), and $\mathbf{f} = (f_1, f_2, f_3)$ is the mass force. We also use the notation $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ for the Laplace operator, $\nabla = (\partial_x, \partial_y, \partial_z)$ for the gradient operator, and $\mathrm{div}\,\mathbf{u} \equiv \nabla\cdot\mathbf{u}$ for the divergence of the vector field $\mathbf{u}$.

1.3. Stokes equations for viscous incompressible fluids

The system of equations governing slow motion of a viscous incompressible fluid consist of the equation of motion and the equation of incompressibility [6–8,21,22]:

$$\begin{aligned}&\mathbf{u}_t = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{u} + \mathbf{f},\\&\mathrm{div}\,\mathbf{u} = 0,\end{aligned} \tag{2}$$

where $\mathbf{u}$ is the fluid velocity, $\mathbf{f}$ is the mass force, ρ is the density, p is the pressure, and ν is the kinematic viscosity.

1.4. Stokes equations for viscous compressible barotropic fluids

The system of equations governing motion of a viscous compressible fluid can be written in the form [23–25]:

$$\begin{aligned}&\rho[\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}] = -\nabla p + \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\,\mathrm{div}\,\mathbf{u} + \rho\mathbf{f},\\&\rho_t + \mathbf{u}\cdot\nabla\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0,\end{aligned} \tag{3}$$

where μ and λ are the coefficients of dynamic viscosity; the rest of the notation is the same as in (2). For barotropic compressible fluids, equations (3) are complemented by a constitutive equation that can be solved with respect to density:

$$\rho = \rho(p), \tag{4}$$

where $\rho(p)$ is a given function.