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An implicit family of time marching procedures with adaptive dissipation control

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ABSTRACT

In this work, an unconditionally stable family of time marching procedures is proposed, in which the time integration parameters of the method are locally computed, considering the properties of the model. Thus, the technique adapts to the characteristics of the problem, enabling enhanced analyses. Here, adaptive dissipation control is explored, allowing spurious modes to be more effectively dissipated. The proposed technique is second-order accurate, *L*-stable, truly self-starting and very easy to implement. Numerical results are presented along the paper, illustrating the good performance of the novel single-step technique.

1. Introduction

There are several applications in various branches of science and engineering regarding time dependent hyperbolic models. Considering these models, analytical transient responses are usually too complex to obtain and they are not available, having numerical techniques usually been applied to find step-by-step approximate solutions, taking into account time integration algorithms.

As referred in Hughes [1], considerable effort has gone into the development of efficient computational methods for the step-by-step integration of hyperbolic models. Although there is no universal consensus, it is generally agreed that for a method to be competitive it should possess the following attributes: (i) unconditional stability when applied to linear problems; (ii) no more than one set of implicit equations should have to be solved at each step; (iii) second-order accuracy; (iv) controllable algorithmic dissipation in the higher modes; (v) self-starting.

When solving hyperbolic models using direct time integration methods, algorithmic damping is useful to control spurious high-frequency oscillations. Algorithmic damping also has been found to be helpful when solving problems that include constraints and some nonlinearities (e.g., contact, nonlinear relations, etc.). Hence, it is desirable for an algorithm to have controllable numerical dissipation in the high frequency range. Typically, physical high-frequency characteristics are modeled inaccurately and these inaccuracies are manifested in the time response as spurious high-frequency oscillations. Thus, the low-frequency responses should be preserved while the high-frequency responses should be damped in a controllable way.

The dissipation can be measured by the spectral radius (ρ), which is defined as the largest magnitude of the eigenvalues of the numerical amplification matrix. According to Wood [2], the curve $\rho \times \Delta t/T$ (where Δt and T stand for the time step and the undamped natural period of the model, respectively) should remain close to the unitary value as long as possible and then

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decrease to about 0.5 and 0.8, as $\Delta t/T$ increases to infinity. The ρ value, as $\Delta t/T \to \infty$, is known as the ultimate spectral radius (ρ_∞). When ρ_∞ approaches zero, asymptotic annihilation occurs, and high frequency responses are eliminated in one time step. The implicit Euler method is the most well-known asymptotically annihilating algorithm. This technique is accurate only to first order and it provides too much dissipation in the low-frequency range, being inappropriate for several problems. The Houbolt method [3] and the Park method [4] are also well-known asymptotically annihilating techniques. However, the Houbolt method is very dissipative in the low-frequency range as well, and the Park method may not be very convenient, since it is equivalent to a six-step linear multi-step method, requiring special start-up procedures.

Enhanced time integration methods introduce parameters that can control the degree of numerical damping, providing more flexibility and effectiveness in the analyses [5]. As examples of these techniques, one can refer to the Newmark method [6], with parameters β and γ; the Hilber–Hughes–Taylor α method (HHT) [7]; the Zhou and Tamma generalized single step single solve (GSSSS) integration algorithm, with parameter ρ_∞ [8], etc. The Newmark family is widely employed, but attributes (iii) and (iv) do not simultaneously apply for this family, since $\gamma = 1/2$ is required for second-order accuracy, which precludes dissipation. Aiming to improve the performance of numerical integration techniques respecting attribute (ii), HHT, generalized-α [9], Wood–Bossak–Zienkiewicz (WBZ) [10] and GSSSS methods were developed considering the so-called "weighted residual" approach. Following this approach, modified expressions of updates and acting forces are substituted into the dynamic equations, improving the integration process. However, the rate of convergence for low-frequency accelerations is worsen by this approach (in comparison with displacements and velocities), requiring an extra solution of equations to eliminate this deficiency (employing direct substitution of updated levels) [8], which dismisses attribute (ii).

A lot of research has been realized in the last decades and several time-marching algorithms with controllable dissipative features are currently available to analyze dynamic models [11–23]. In this work, a novel family of implicit time marching techniques is proposed, in which adaptive dissipation control is introduced. In this new approach, all the five attributes above described are retained, and the proposed technique is very easy to implement. The method stands as an asymptotically annihilating algorithm, being very effective to accurately compute low-frequency responses and to dissipate high-frequency modes. In addition, the technique is only based on displacements–velocities relations, requiring no computation of accelerations. Thus, if required, accelerations can be post-processed by any suitable technique. As a consequence, the family is very simple and truly self-starting, eliminating any kind of cumbersome initial procedure, such as the computation of multi-step initial values or initial accelerations. An equivalent approach has been considered in reference [22], taking into account explicit conditionally stable time marching techniques. Here, unconditionally stable procedures are discussed, allowing larger time steps to be considered, as well as enabling a larger range of high-frequency modes to be properly dissipated (the method is *L*-stable). Numerical results are described along the manuscript, illustrating the good performance of the new technique, as well as its potentialities.

2. Governing equations and time integration strategy

A linear dynamic model is governed by the following system of equations [24]:

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{F}(t), \tag{1}$$

where **M, C** and **K** stand for the mass, damping and stiffness matrix, respectively, $\mathbf{F}(t)$ stands for the force vector and $\mathbf{U}(t)$, $\dot{\mathbf{U}}(t)$ and $\ddot{\mathbf{U}}(t)$ stand for the displacement, velocity and acceleration vector, respectively. The initial conditions of the model are given by:

$$\mathbf{U}^0 = \mathbf{U}(0), \tag{2a}$$

$$\dot{\mathbf{U}}^0 = \dot{\mathbf{U}}(0), \tag{2b}$$

where $\mathbf{U}^0$ and $\dot{\mathbf{U}}^0$ represent initial displacement and velocity vectors, respectively.

By considering a time-step Δt, each governing equation of the model may be integrated as:

$$\sum_j [\mathrm{M}_{ij} \int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \ddot{\mathrm{U}}_j(\tau) d\tau + \mathrm{C}_{ij} \int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \dot{\mathrm{U}}_j(\tau) d\tau + \mathrm{K}_{ij} \int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \mathrm{U}_j(\tau) d\tau] = \int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \mathrm{F}_i(\tau) d\tau, \tag{3}$$

where the integrals in the l.h.s. may be approximated by:

$$\Im_{\ddot{\mathrm{U}}_j}^{n+\frac{1}{2}} = \int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \ddot{\mathrm{U}}_j(\tau) d\tau \approx \dot{\mathrm{U}}_j^{n+1} - \dot{\mathrm{U}}_j^n, \tag{4a}$$

$$\Im_{\dot{\mathrm{U}}_j}^{n+\frac{1}{2}} = \int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \dot{\mathrm{U}}_j(\tau) d\tau \approx \mathrm{U}_j^{n+1} - \mathrm{U}_j^n, \tag{4b}$$

$$\Im_{\mathrm{U}_j}^{n+\frac{1}{2}} = \int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \mathrm{U}_j(\tau) d\tau \approx \Delta t\, \mathrm{U}_j^n + \tfrac{1}{2}\alpha 2_i \Delta t^2 \dot{\mathrm{U}}_j^n + \tfrac{1}{2}\alpha 1_i \Delta t^2 \dot{\mathrm{U}}_j^{n+1}. \tag{4c}$$