Applied Mathematical Modelling 40 (2016) 2291–2300

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Extremal principal eigenvalue of the bi-Laplacian operator

S.A. Mohammadi [a,*], F. Bahrami [b]

[a] Department of Mathematics, College of Sciences, Yasouj University, Yasouj 75918-74934, Iran
[b] Faculty of Mathematical Sciences, University of Tabriz, 29 Bahman St., Tabriz 51665-163, Iran

ARTICLE INFO

Article history:
Received 18 June 2014
Revised 1 June 2015
Accepted 22 September 2015
Available online 9 October 2015

Keywords:
Bi-Laplacian
Eigenvalue optimization
Rearrangement
Non-homogeneous plates

ABSTRACT

In this paper we propose two numerical algorithms to derive the extremal principal eigenvalue of the bi-Laplacian operator under Navier boundary conditions or Dirichlet boundary conditions. Consider a non-homogeneous hinged or clamped plate Ω, the algorithms converge to the density functions on Ω which yield the maximum or minimum basic frequency of the plate.

1. Introduction

Eigenvalue problems for elliptic partial differential equations have many applications in engineering and applied sciences and these problems have been intensively attractive to mathematicians in the past decades [14].

This paper is concerned with a fourth-order elliptic eigenvalue problem modeling the vibration of a non-homogeneous plate Ω which is either hinged or clamped along the boundary $\partial\Omega$. Several materials with m different kinds of densities $0 < c_1 < c_2 < \cdots < c_m$ are given where the area of the domain with density c_i is $S_i > 0$, $i = 1 \ldots m$. The problem involves geometrical constraints that can be described as $\sum_{i=1}^m S_i$ should be equal to the area of Ω. We investigate the location of these materials throughout Ω in order to optimize the basic frequency in the vibration of the corresponding plate.

Motivated by the above explanation, we introduce the mathematical equations governing the structure and associated optimization problems. Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and let $\rho_0(x) = c_1\chi_{D_1} + \cdots + c_m\chi_{D_m}$, the density function, be a measurable function such that $|D_i| = S_i > 0$, $(i = 1 \ldots m)$ and $\sum_{i=1}^m S_i = |\Omega|$ where $|.|$ stands for Lebesgue measure. Define $\mathcal{P}$ as the family of all measurable functions which are rearrangement of ρ_0. For $\rho \in \mathcal{P}$, consider eigenvalue problems

$$\Delta^2 u = \lambda\rho u, \quad \text{in} \quad \Omega, \quad u = 0, \ \Delta u = 0, \qquad \text{on} \quad \partial\Omega, \tag{1.1}$$

$$\Delta^2 v = \Lambda\rho v, \quad \text{in} \quad \Omega, \quad v = 0, \ \frac{\partial v}{\partial n} = 0, \qquad \text{on} \quad \partial\Omega, \tag{1.2}$$

where $\lambda = \lambda_\rho$, $\Lambda = \Lambda_\rho$ are the first eigenvalues or the basic frequencies and $u = u(x)$, $v = v(x)$ are the corresponding eigenfunctions or the lateral displacements. The operator Δ^2 stands for usual bi-Laplacian, that is $\Delta^2 u = \Delta(\Delta u)$. The principal eigenvalue λ of problem (1.1) is obtained by minimizing the associate Rayleigh quotient

* Corresponding author. Tel.: +98 9132385984.
E-mail addresses: mohammadi@yu.ac.ir (S.A. Mohammadi), fbahram@tabrizu.ac.ir (F. Bahrami).

http://dx.doi.org/10.1016/j.apm.2015.09.058

$$\lambda = \inf\left\{ \frac{\int_\Omega (\Delta w)^2 dx}{\int_\Omega \rho w^2 dx} : w \in H^2(\Omega) \cap H_0^1(\Omega), w \neq 0 \right\}, \tag{1.3}$$

and the first eigenvalue Λ of problem (1.2) is obtained by minimizing the associate Rayleigh quotient

$$\Lambda = \inf\left\{ \frac{\int_\Omega (\Delta w)^2 dx}{\int_\Omega \rho w^2 dx} : w \in H_0^2(\Omega), w \neq 0 \right\}, \tag{1.4}$$

where it is well known [24] that the infimum is attained in both cases. By regularity results the solutions to problems (1.1) and (1.2) belongto $H^4_{loc}(\Omega)$ and these equations hold a.e. in Ω, [1].

To determine the system's profile which gives the maximum and minimum principal eigenvalues, Cuccu *et al.* have verified the following optimization problems

$$\max_{\rho\in\mathcal{P}} \lambda_\rho, \tag{1.5}$$

$$\min_{\rho\in\mathcal{P}} \lambda_\rho, \tag{1.6}$$

$$\max_{\rho\in\mathcal{P}} \Lambda_\rho, \tag{1.7}$$

$$\min_{\rho\in\mathcal{P}} \Lambda_\rho, \tag{1.8}$$

in [4,9,10]. The existence of solutions for problems (1.5)–(1.6) and (1.8) have been proved for general domain Ω. But, the existence of a solution for problem (1.7) has been established when Ω is a positivity preserving domain for $\Delta^2 u$ under homogeneous Dirichlet boundary conditions. For instance, the ball is a domain that enjoys such a property. In spite of these existence results, the precise identifications of the maximums and minimums were found only in case Ω is a ball.

In eigenvalue optimization for elliptic partial differential equations, one of challenging mathematical problems after the problem of existence is an exact formula of the optimizer or optimal shape design. Most papers in this field answered this question just in case Ω is a ball. For other domains qualitative properties of solutions were investigated and partial answers were given [6–8,11,20,21]. From the physical point of view, it is important to know the shape of the optimal density functions in case Ω is not a ball.

This class of problems is difficult to solve because of the lack of the topology information of the optimal shape. There must be numerical approaches to determine the optimal shape design. The mostly used methods now are the homogenization method [2] and the level set method [23]. The level set method is well known for its capability to handle topological changes, such as breaking one component into several, merging several components into one and forming sharp corners. This approach has been applied to the study of extremum problems of eigenvalues of inhomogeneous structures including the identification of composite membranes with extremum eigenvalues [13,22,25], design of composite materials with a desired spectral gap or maximal spectral gap [16], finding optical devices that have a high quality factor [17] and principle eigenvalue optimization in population biology [15].

Recently, Kao and Su [18] proposed an efficient rearrangement algorithm based on the Rayleigh quotient formulation of eigenvalues. They have solved minimization and maximization problems for the kth eigenvalue ($k \geq 1$) and maximization of spectrum ratios of the second order elliptic differential operator in $\mathbb{R}^2$. We extend the approach to solve a fourth order partial differential equation. Most of the previous results are for second order operators with Dirichlet boundary conditions and the geometric constraint have been considered for $m = 2$. Here we study a partial differential equation with different boundary conditions and we develop our algorithms for general $m \geq 2$. It is common in the literature that Ω have been considered as a rectangle [19]. To show the capacity and efficiency of the numerical method, we apply it to solve problems (1.5)–(1.8) when Ω has a more complicated geometrical structure than a rectangle. The algorithms start from a given density ρ_0 and generate monotone sequences of the basic frequencies. In order to guarantee the monotonicity of the sequences, we introduce an acceptance and rejection method. It is established that the derived monotone sequences are convergent.

2. Analytical results and numerical methods

In this section we describe the analytic results which our algorithms are based upon them. Then, we propose algorithms to derive the solutions of problems (1.5)–(1.8) respectively.

The algorithms strongly based on the Rayliegh quotients in formulas (1.3) and (1.4). Such algorithms have been applied successfully to minimize eigenvalues of some second order elliptic operators [6,18]. They rely on the variational formulation of the eigenvalues and use level sets of the eigenfunctions or gradient of them. Employing the level sets of the eigenfunctions, we need some results of the rearrangement theory with an eye on our problem, see the Appendix A.

Throughout this paper we shall write increasing instead of non-decreasing, and decreasing instead of non-increasing.

2.1. Minimization problems

For the minimization problems (1.6) and (1.8), we start from a given initial density functions ρ_0 and extract new density functions ρ_1 using the eigenfunction of Eqs. (1.1)–(1.2) such that the first eigenvalues are decreased, i.e.