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Goal-oriented error estimation for beams on elastic foundation with double shear effect

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ABSTRACT

In this paper, goal-oriented error estimation for Timoshenko beams on Pasternak foundation, which involves double shear effect, is performed. The constitutive relation error (CRE) estimation is used in finite element analysis (FEA) to acquire strict bounds on quantities of interest. Due to the coupling of the displacement field and the internal force field in the equilibrium equations of the beam, the prolongation condition for construction of the admissible internal force field, a pillar of the CRE estimation, is not directly applicable. To overcome this difficulty, an auxiliary approximate problem whose stress solution enables the CRE estimation to proceed is introduced. Thereafter, strict bounds of outputs for the beam are obtained by dual analysis to which a significant adjunct is the circumvention of shear locking in low-order finite element analysis. Numerical results are presented to validate the strict bounding properties for quantities of beams on elastic foundation and accurate estimation of displacement quantities that is impervious to shear locking.

1. Introduction

A beam on an elastic foundation [1] is a classical problem and often encountered in practice. Under the assumption that the elastic foundation is homogeneous and isotropic, plenty of models have been established [2–6] for simulating the action of soil media. Among the models, the Pasternak foundation model [4] appears to be an important one, taking into account both contact pressure and shear action of the soil. Often the transverse shear deformation of a short beam is not negligible. Timoshenko beam model [7], which allows for transverse shear deformation, is used exclusively in the present work to treat beams on Pasternak foundation involving double shear effect.

The finite element method (FEM) has been widely used in engineering design to make critical decisions. In order to develop confidence in the decisions, a research topic of controlling the quality or the error of numerical simulations, termed *model verification*, has been intensively studied for more than three decades. Among various sources of error, the discretization error is predominant and controllable [8]. Several families of *a posteriori* error estimators [9–11] have been presented, such as explicit error estimators [12], implicit error estimators [13], recovery-based error estimators [14], hierarchical estimators [15], constitutive relation error (CRE) estimators [16], etc.

In finite element analysis, it is frequently the case that *a posteriori* finite element error analysis is focused on *goal-oriented error estimation*, which assesses the errors in specific quantities of interest, such as local values of stresses, displacements etc. To do this, adjoint/dual-based techniques are used to represent and estimate the errors in solution outputs, which have been systematically reviewed in [17–20]. Goal-oriented error estimation methods have been developed since mid-1990s (see, for

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example, [21–25]). Several specific estimates, such as the adjoint-weighted residual method [17,20], the energy norm based estimates [26], the Green's function decomposition method [27], the strict-bounding approach based on Lagrangian formulation [28], the CRE-based error estimation [25], have been proposed and applied in solutions of Poisson's equation, linear and non-linear static problems in solid mechanics, eigenvalue problems, time-dependent problems, non-trivial problems of CFD and etc [29,30]. Among the available techniques, the CRE-based error estimation is a one to guarantee strict bounds of quantities. The strict bounding property, together with its advantage of wide applicability [31–38], makes the CRE stand out for goal-oriented error estimation.

During the CRE estimation, an equilibrated stress (internal force) field is required which can be constructed from the finite element (FE) results through a prolongation condition [8,39]. For Timoshenko beams on Pasternak foundation, however, the equilibrium equations involve not only internal forces, but also an unknown reaction force due to the existence of foundation. The coupling of stress and the unknown reaction force in the equilibrium equations of the beam hinders direct application of the prolongation condition which normally caters for a problem with equilibrium equations expressed in terms of stresses. In this paper, an expedient way to overcome this difficulty is developed by means of the introduction of an auxiliary approximate problem whose equilibrium equations change into a form involving only stresses. In doing so, the CRE estimation is revitalized and strict bounds of quantities for beams on the foundation become obtainable.

It is well-known that *shear locking* may occur in low-order displacement-type finite element analysis of bending-dominant Timoshenko beams, resulting in unexpected smaller magnitude of displacements [7]. Modified solutions of the displacements of Timoshenko beams can be obtained by means of dual analysis with highly enhanced accuracy to meet design requirements. This is another achievement of the paper.

Following the introduction, the Timoshenko beam on Pasternak foundation and its finite element formulation are introduced in Section 2. In Section 3, the concept and the strict upper bound property of the CRE estimation for beams on elastic foundation are elaborated. The strict upper and lower bounds of various quantities for the beam are obtained by goal-oriented error estimation in Section 4, and discussion on overcoming shear locking is conducted in Section 5. In Section 6, numerical results are presented to validate the goal-oriented error estimation for the beam, and conclusions are drawn in Section 7.

2. A Timoshenko beam on Pasternak foundation

2.1. Primal problem

Consider a Timoshenko beam on Pasternak foundation, which occupies an interval $X = (a, b)(a < b)$, with $x = a$ and $x = b$ being the two end-points of the interval. As shown in Fig. 1, the flexural stiffness and shear stiffness of the beam are denoted by EI and κGA; the subgrade reaction coefficient of the foundation and modulus of the shear layer are designated as $k_f(x)$ and $G_f(x)(k_f, G_f > 0)$. Distributed load $q(x) \in L_2(X)$ and concentrated loads are applied on the beam. For the Timoshenko beam, average shear strain is expressed by normal rotational angle θ and deflection w as $\gamma = \theta + \frac{dw}{dx}$. Two internal forces, bending moment M and shear force V, are taken into consideration. Let T represent the reaction force from the elastic foundation, enjoying the constitutive law invoked by the elastic foundation $T = k_f w - \frac{d}{dx}\left(G_f \frac{dw}{dx}\right)$. Thus the strong form of the primal problem can be described as:Find the displacement fields $\boldsymbol{u} := \begin{Bmatrix} w \\ \theta \end{Bmatrix}$ and the internal forces $\boldsymbol{s} := \begin{Bmatrix} V \\ M \end{Bmatrix}$, which satisfy:

- *Kinematic constraints*

$$\begin{Bmatrix} w \\ \theta \end{Bmatrix} \in \mathcal{U}_p := \left\{ \begin{Bmatrix} w \\ \theta \end{Bmatrix} \in [H^1(X)]^2 : w \text{ and } \theta \text{ satisfy Dirichlet boundary conditions} \right\}, \tag{1}$$

where the space $\mathcal{U}_p$ is called the primal space to which the displacement fields $\begin{Bmatrix} w \\ \theta \end{Bmatrix}$ belong;

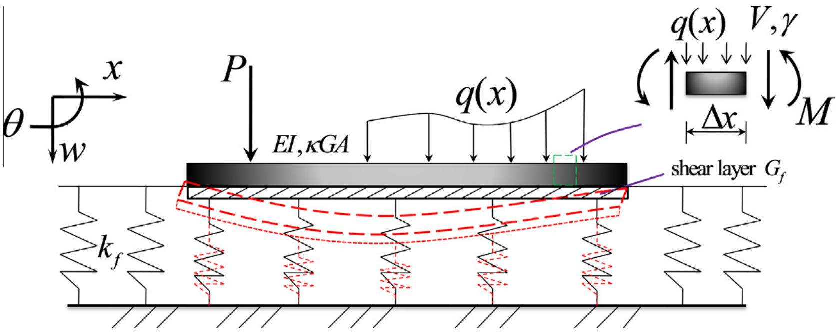

Fig. 1. A Timoshenko beam on Pasternak foundation.