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Short communication

A linear algebraic approach for the computation of sums of Erlang random variables

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ABSTRACT

We propose a matrix analysis approach to analytically provide the cumulative distribution function of the sum of independent Erlang random variables. This reduces to the characterization of the exponential of the involved generator matrix. We propose a particular basis of vectors in which we write the generator matrix. We find, in the new basis, a Jordan–Chevalley decomposition allowing to simplify the calculation of the exponential of the generator matrix. This is a simpler alternative approach to the existing ones in the literature.

1. Introduction

Many situations in service and manufacturing service systems involve the computation of the sum of independent exponential random variables. Examples include healthcare or production systems with different stages in series, system reliability with exponentially distributed components lifetimes, and wireless mobile systems with cooperative diversity schemes. This summation arises also in the transient analysis of Markovian queueing systems, and in general, semi-Markov processes.

We consider the general case of a hypoexponential distribution defined as the sum of n independent Erlang distributions, for $n \in \mathbb{N}$. An Erlang distribution is defined by two parameters, a number of i.i.d. exponential stages and a rate per stage. Thus, the general hypoexponential distribution is completely defined by the couples of parameters (λ_i, k_i) for $i = 1, \ldots, n$. Each couple (λ_i, k_i) defines an Erlang distribution ($\lambda_i \in \mathbb{R}, k_i \in \mathbb{N}$), and the rates λ_i for $i = 1, \ldots, n$ are all distinct. We denote by $K_i = k_1 + k_2 + \ldots + k_i$ for $i = 1, \ldots, n$ and use the convention $K_0 = k_0 = 0$. The cumulative distribution function (cdf) of the hypoexponential distribution is then given by

$$F(x) = 1 - \boldsymbol{\alpha} e^{xM}\mathbf{1}, \tag{1}$$

for $x \geqslant 0$, where $\mathbf{1}$ is a column vector of size K_n with ones everywhere, $\boldsymbol{\alpha}$ is a line vector of size K_n and is given by $\boldsymbol{\alpha} = (1, 0, \ldots, 0)$, and $e^{(\cdot)}$ denotes the exponential operator. The generator square matrix M of size $K_n \times K_n$ is defined by the coefficients $m_{i,j}$ for $i, j \in \{1, \ldots, K_n\}$. We have $m_{j,j} = -\lambda_i$ and $m_{j,j+1} = \lambda_i$, for $K_{i-1} + 1 \leqslant j \leqslant K_i$ and $i = 1, \ldots, n$. All remaining coefficients of M are zero. We thus may write

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$$M = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & -\lambda_1 & \lambda_1 & \ddots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 & -\lambda_1 & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & -\lambda_2 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & -\lambda_n & \lambda_n \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & -\lambda_n \end{pmatrix}. \tag{2}$$

Scheuer [1] provides a formula for $F(.)$ that involves high order derivatives of products of multiple functions. The formula is however hard to compute numerically. Amari and Misra [2] propose a simplification of Scheuer[1]'s formula using Laplace transforms and multi-function generalization of the Lebnitz rule for higher order derivatives of products of two functions. For a particular case with constraints on the values of the λ_is, Van Khuong and Kong [3] provide the probability distribution function by inverting its Fourier transform. Using the Wilk's integral representation of the distribution of the product of independent beta random variables, Favaro and Walker [4] provide an alternative formula for $F(\cdot)$. We also refer the reader for more details to the review by Nadarajah [5].

In this paper, we propose an alternative simple approach to analytically derive the cdf of $F(\cdot)$. It is based on a linear algebraic matrix analysis. The structure of the approach is as follows. We first obtain some particular eigenvectors of the generator matrix M. These are next used to construct a new basis of vectors. The new basis allows to find the Jordan–Chevalley decomposition of M into a sum of two commutative linear operators, a diagonal one and a nilpotent one. The exponential of the matrix M then simply follows by inverting the new basis matrix using the Cayley–Hamilton theorem, which leads to the cdf of $F(\cdot)$.

2. The result

Lemma 1 provides the eigenvalues of the matrix M, and one eigenvector associated to each eigenvalue.

Lemma 1. *The eigenvalues of M are $-\lambda_i$ for $i = 1, \dots, n$. An eigenvector of size K_n associated to $-\lambda_i$ is the column vector u_i, where the coefficients of u_i, denoted by $u_{i,l}$ for $1 \leqslant l \leqslant K_n$, are given by*

$$\begin{cases} u_{i,l} = 1 & , l = K_{i-1} + 1, \\ u_{i,l} = 0 & , l > K_{i-1} + 1, \\ u_{i,l} = \left(\frac{\lambda_{i-1}}{\lambda_{i-1} - \lambda_i}\right)^{K_{i-1} - l + 1} & , K_{i-2} + 1 \leqslant l \leqslant K_{i-1}, \\ u_{i,l} = \prod_{j=1}^{i-(m+1)} \left(\frac{\lambda_{i-j}}{\lambda_{i-j} - \lambda_i}\right)^{k_{i-j}} \left(\frac{\lambda_m}{\lambda_m - \lambda_i}\right)^{K_m - l + 1} & , K_{m-1} + 1 \leqslant l \leqslant K_m \text{ and } 0 \leqslant m < i - 1. \end{cases}$$

Proof. Since M is a triangular matrix, its eigenvalues are its diagonal coefficients, i.e., $-\lambda_i$ for $i = 0, \dots, n$. For $0 \leqslant i = 0, \dots, n$, consider the column vector u_i defined with its coefficients $u_{i,l}$ $(1 \leqslant l \leqslant K_n)$, where $u_{i,K_{i-1}+1} = 1, u_{i,l} = 0$ for $l > K_{i-1} + 1, u_{i,l} = \left(\frac{\lambda_{i-1}}{\lambda_{i-1} - \lambda_i}\right)^{K_{i-1} - l + 1}$ for $K_{i-2} + 1 \leqslant l \leqslant K_{i-1}$, and $u_{i,l} = \prod_{j=1}^{i-(m+1)} \left(\frac{\lambda_{i-j}}{\lambda_{i-j} - \lambda_i}\right)^{k_{i-j}} \left(\frac{\lambda_m}{\lambda_m - \lambda_i}\right)^{K_m - l + 1}$, for $K_{m-1} + 1 \leqslant l \leqslant K_m$ and $0 \leqslant m < i - 1$. We now define, for $0 \leqslant i = 0, \dots, n, v_i$ as $v_i = M u_i$ and we denote its coefficients by $v_{i,l}$, for $1 \leqslant l \leqslant K_n$. Consider $1 \leqslant j \leqslant n$. For $K_{j-1} + 1 \leqslant l \leqslant K_j$ and $l \neq K_n$, we have

$$v_{i,l} = -\lambda_j u_{i,l} + \lambda_j u_{i,l+1}, \tag{3}$$

and $v_{i,K_n} = -\lambda_n u_{i,K_n}$. Since $u_{i,l} = 0$ and $u_{i,K_{i-1}+1} = 1$ for $l > K_{i-1} + 1$, we deduce from Eq. (3) that $v_{i,l} = 0$ for $l > K_{i-1} + 1$ and $v_{i,K_{i-1}+1} = -\lambda_i$. For $K_{i-2} + 1 \leqslant l \leqslant K_{i-1}$ we have $u_{i,l} = \left(\frac{\lambda_{i-1}}{\lambda_{i-1} - \lambda_i}\right)^{K_{i-1} - l + 1}$, and for $K_{m-1} + 1 \leqslant l \leqslant K_m$ and $0 \leqslant m < i - 1$ we have $u_{i,l} = \prod_{j=1}^{i-(m+1)} \left(\frac{\lambda_{i-j}}{\lambda_{i-j} - \lambda_i}\right)^{k_{i-j}} \left(\frac{\lambda_m}{\lambda_m - \lambda_i}\right)^{K_m - l + 1}$. Eq. (3) therefore leads to $v_{i,l} = -\lambda_{i-1} \left(\frac{\lambda_{i-1}}{\lambda_{i-1} - \lambda_i}\right)^{K_{i-1} - l + 1} + \lambda_{i-1} \left(\frac{\lambda_{i-1}}{\lambda_{i-1} - \lambda_i}\right)^{K_{i-1} - (l+1) + 1} = -\lambda_i \left(\frac{\lambda_{i-1}}{\lambda_{i-1} - \lambda_i}\right)^{K_{i-1} - l + 1}$, for $K_{i-2} + 1 \leqslant l \leqslant K_{i-1}$. We also obtain $v_{i,l} = -\lambda_i \prod_{j=1}^{i-(m+1)} \left(\frac{\lambda_{i-j}}{\lambda_{i-j} - \lambda_i}\right)^{k_{i-j}} \left(\frac{\lambda_m}{\lambda_m - \lambda_i}\right)^{K_m - l + 1}$, for $K_{m-1} + 1 \leqslant l \leqslant K_m$ and $0 \leqslant m < i - 1$. This proves that u is an eigenvector associated to the eigenvalue $-\lambda_i$, for $i = 1, \dots, n$, and finishes the proof of the lemma. $\square$