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Two-level iteration penalty methods for the incompressible flows ☆

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ABSTRACT

In this article, we present a new iteration penalty method for incompressible flows based on the iteration of pressure with a factor of penalty parameter, which was first developed for Stokes flows by Cheng and Abdul (2006) [14]. The stability and error estimates of numerical solutions in some norms are derived for this one-level method. Then, combining the techniques of two-level method and linearization with respect to the nonlinear convective term, we propose two-level Stokes/Oseen/Newton iteration penalty methods corresponding to three different linearization method, and show the stability and error estimates of these three methods. Finally, some numerical tests are given to demonstrate the effect of penalty parameter and the efficiency of the new methods.

1. Introduction

In this paper, we consider a two-level iteration penalty method for the incompressible flows which are governed by the incompressible Navier–Stokes equations

$$\begin{cases} -\mu\Delta u + (u\cdot\nabla)u - \nabla p = f, & \text{in } \Omega, \\ \text{div} u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ assumed to have a Lipschitz continuous boundary $\partial\Omega$. $\mu > 0$ represents the viscosity coefficient. $u = (u_1(x), u_2(x))$ denotes the velocity vector, $p = p(x)$ the pressure, $f = (f_1(x), f_2(x))$ the prescribed body force vector. The solenoidal condition $\text{div}u = 0$ means that the flows are incompressible.

The development of appropriate mixed finite element approximations is a key component in the search for efficient techniques for solving the problem (1.1) quickly and efficiently. Roughly speaking, there exist two main difficulties. One is the nonlinear term $(u\cdot\nabla)u$, which can be processed by the linearization methods such as the Newton iteration method [1], or the two-level method [2–9]. The other is that the velocity and the pressure are coupled by the solenoidal condition. The popular technique to overcome this difficulty is to relax the solenoidal condition in an appropriate method, resulting in a

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pesudo-compressible system, such as the penalty method and the artificial compressible method [10]. Recently, using the Taylor–Hood element ($P_2 - P_1$ triangular element), Li and An [11] studied two-level penalty finite element methods for Navier–Stokes equations with nonlinear slip boundary conditions, where the main results can be extended to the problem (1.1). Denote $(u_\varepsilon^h, p_\varepsilon^h)$ the two-level penalty finite element approximation solution to $(u,p) \in (H^3(\Omega)^2, H^2(\Omega))$. The error estimate derived in [11] is

$$||u - u_\varepsilon^h||_1 + ||p - p_\varepsilon^h|| \leqslant c(\varepsilon + h^2 + H^3), \tag{1.2}$$

where $\varepsilon > 0$ is small, h and H are the fine mesh size and coarse mesh size, respectively, and satisfy $h < H < 1$. $c > 0$ is independent of ε, h and H. Thus, it suggests that ε depends on h, i.e. $\varepsilon = O(h^2)$, to yield an accurate approximation. However, the condition number of the numerical discretization of two-level penalty methods is $O(\varepsilon^{-1}h^{-2})$, which will result in a very ill-conditioned problem when mesh size $h \to 0$.

In this paper, we combine the iteration penalty method with the two-level method to solve the numerical solution to (1.1). The iterative penalty method was first introduced by Cheng [12] for the Stokes equations and further used to solve the pure Neumann problem [13] and the Navier–Stokes equations with nonlinear slip boundary conditions [14]. This iteration penalty method allows us to use a "not very small" penalty parameter ε. Our two-level iteration penalty methods can be described as follows. The first step and the second step are required to solve a small Navier–Stokes equations on the coarse mesh in terms of the iteration penalty method [12,14]. The third step is required to solve a large linearization problem on the fine mesh in terms of Stokes iteration, Oseen iteration or Newton iteration, respectively. We prove that these two-level iteration penalty finite element solutions $(u_{\varepsilon h}, p_{\varepsilon h})$ are of the following error estimate

$$||u - u_{\varepsilon h}||_1 + ||p - p_{\varepsilon h}|| \leqslant \begin{cases} c(h^2 + H^3 + \varepsilon H^2 + \varepsilon^{k+1}) & \text{Stokes/Oseen methods} \\ c(h^2 + H^4 + \varepsilon H^2 + \varepsilon^{k+2}) & \text{Newton method} \end{cases} \tag{1.3}$$

for any positive integer k. Thus, if we choose $\varepsilon = O(H) = O(h^{2/3})$, then (1.3) is of the optimal convergence rate of same order as the usual Galerkin finite element method. Therefore, compared to the two-level penalty method in [11], our iteration penalty method allows that ε is not very small. Moreover, combining with two-level methods, our method we study in this paper can save a large amount of computational time and is an efficient numerical method for solving the numerical solution to the problem (1.1).

2. Preliminary

In what follows, we employ the standard notation $H^l(\Omega)$ (or $H^l(\Omega)^2$) and $||\cdot||_l, l \geqslant 0$, for the Sobolev spaces of all functions having square integrable derivatives up to order l in Ω and the standard Sobolev norm. When $l = 0$, we shall write $L^2(\Omega)$ (or $L^2(\Omega)^2$) and $||\cdot||$ instead of $H^0(\Omega)$ (or $H^0(\Omega)^2$) and $||\cdot||_0$, respectively. Let X be a Banach space. Denote by X' the dual space of X and by $<\cdot,\cdot>_X$ the dual product between X and X'. The dual norm $||\cdot||_{X'}$ is defined by $||v||_{X'} = \sup_{w\in X} \frac{<v,w>_X}{||w||_X}$.

For the mathematical setting, we introduce the following spaces:

$$V = H_0^1(\Omega)^2, \quad M = L_0^2(\Omega) = \left\{q \in L^2(\Omega);\ \int_\Omega q dx = 0\right\}.$$

The space V is equipped with the norm

$$||v||_V = \left(\int_\Omega |\nabla v|^2 dx\right)^{1/2}.$$

It is well known that $||v||_V$ is equivalent to $||v||_1$. Introduce two bilinear forms

$$a(u,v) = \mu \int_\Omega \nabla u \cdot \nabla v dx, \quad \forall\ u,\ v \in V,$$
$$d(v,q) = \int_\Omega q \operatorname{div} v dx, \quad \forall\ v \in V,\ q \in M,$$

and a trilinear form

$$b(u,v,w) = \int_\Omega (u\cdot\nabla)v\cdot w dx - \frac{1}{2}\int_\Omega \operatorname{div} u v \cdot w dx = \frac{1}{2}\int_\Omega (u\cdot\nabla)v\cdot w dx - \frac{1}{2}\int_\Omega (u\cdot\nabla)w\cdot v dx.$$

It is easy to verify that this trilinear form satisfies the following important properties [7]:

$$b(u,v,w) = -b(u,w,v), \tag{2.1}$$
$$b(u,v,w) \leqslant N||u||_V||v||_V||w||_V, \tag{2.2}$$
$$b(u,v,w) \leqslant \frac{N}{2}||u||^{1/2}||u||_V^{1/2}(||v||_V||w||^{1/2}||w||_V^{1/2} + ||w||_V||v||^{1/2}||v||_V^{1/2}), \tag{2.3}$$