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Generalization of the multi-scale finite element method to plane elasticity problems

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ABSTRACT

In this paper, according to the governing differential equations of problem, the theory to construct the shape functions in the multi-scale finite element method is established for plane elasticity problems. An approach is then suggested to numerically solve the shape functions via the corresponding homogeneous governing equations on an element level. The linear, quadratic and cubic shape functions are finally obtained by prescribing the appropriate boundary conditions. Typical numerical experiments are conducted, including bending of a homogeneous beam, bending of a beam with voids, as well as bending of a beam with a random material distribution and with an oscillatory material property. The current work shows that the multi-scale finite element method has a prominent advantage in solution efficiency even for classic problems, and therefore can be implemented on a considerably coarse mesh for problems with complex microstructures, as well as for large scale problems to effectively save the solution cost.

1. Introduction

In recent decades, the conventional finite element method (CFEM) was greatly improved from many aspects. Typical examples include the numerical manifold method (NMM) proposed by Shi in 1991 [1], the generalized finite element method (GFEM) proposed by Babuska et al. in 1996 [2] and the extended finite element method (XFEM) proposed by Moes et al. in 1999 [3]. Although the shape functions are enriched in these methods for different purposes, the polynomial shape functions are still retained as the essential. In consequence, the shape functions are expressed either in terms of global area coordinates for triangular elements or in terms of local parent coordinates for quadrilateral elements (e.g. see [4,5]). Because the shape functions are a priori determined which are independent of problems to be solved, the solution accuracy and efficiency are often case dependent.

In virtue of this situation, for a problem with rough coefficients, Babuska et al. [6] proposed the idea that the basis functions adapted the specific problem, which was later developed by Hou et al. [7] to be the multi-scale finite element method (MsFEM). The MsFEM possesses numerical shape functions by solving a given problem, and therefore can, on one hand, capture the effect of microstructures on the macroscopic properties, and, on the other hand, obtain the response in a micro-scale via its shape functions. The convergence and the scale effect of the MsFEM have been verified from the mathematical

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viewpoint [8,9]. The MsFEM was then successfully applied to many problems such as heat transfer in composite materials and porous media [7], the singularly perturbed convection–diffusion equation [10], the three-dimensional incompressible Navier–Stokes equations [11], and flow and transport equations [12]. Recently, the MsFEM was further developed to be the Generalized MsFEM (GMsFEM) to perform multiscale simulations for problems without scale separation over a complex input space [13].

In recent years, the MsFEM was developed to be the extended MsFEM (EMsFEM) by taking the Poisson's effect among different directions [14] to reduce the error in interpolating the vector field [15]. The EMsFEM was applied to more wide physical problems such as thermal conduction simulation in granular materials [16], elasto-plastic analysis [14,17], active response [18], 2D elastostatic analysis of heterogeneous materials [19], and 2D large displacement – small strain analysis of heterogeneous materials [20].

Motivated by Hou et al. [7] and Zhang et al. [14], the MsFEM is further generalized in this paper. In Section 2, through dissecting the plane elasticity problem, the theory is first established to construct the shape functions, and the boundary conditions are then discussed for the shape functions with appropriate orders. In Section 3, an effective approach is suggested to numerically solve the shape functions, and the partition of unity property is verified. In Section 4, the shape functions are applied to some examples to validate the accuracy and the efficiency. The concluding remarks are finally made in Section 5.

2. Basic theory

In this paper, plane elasticity problems are considered. In this case, the governing differential equations in stress form is

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } \Omega, \tag{1}$$

where σ_{ij} are the components of stress tensor and $f_i(x, y)$ are the components of body force vector. The subscript, j denotes the partial differentiation with respect to coordinate x if $j = 1$ or y if $j = 2$. In addition, as a well-defined problem, over the whole boundary $\Gamma = \partial\Omega = \Gamma_u \cup \Gamma_t$, the essential boundary conditions and natural boundary conditions are prescribed respectively on boundary Γ_u and Γ_t, respectively.

After invoking the Hooke's law and the linear strain–displacement relations, Eq. (1) are eventually rewritten in terms of displacement field (u, v) as the following Navier equations [21]

$$\begin{cases} \nabla\cdot(\mu\nabla u) + \frac{\partial}{\partial x}[(\lambda+\mu)\theta] + f_x = 0, \\ \nabla\cdot(\mu\nabla v) + \frac{\partial}{\partial y}[(\lambda+\mu)\theta] + f_y = 0, \end{cases} \tag{2}$$

where λ and μ are the Lame constants, and may vary with position (x, y). Here, we assume that the material is isotropic but may be inhomogeneous and therefore the two constants vary with material points. $\theta = \partial u/\partial x + \partial v/\partial y$ is the displacement divergence (volume strain) in two dimensions.

For a linear problem, due to the superposition principle [21,22] and considering the basis feature of shape functions in the finite element method, the homogeneous form of Eq. (2) is fundamental, i.e.

$$\begin{cases} \nabla\cdot(\mu\nabla u) + \frac{\partial}{\partial x}[(\lambda+\mu)\theta] = 0, \\ \nabla\cdot(\mu\nabla v) + \frac{\partial}{\partial y}[(\lambda+\mu)\theta] = 0. \end{cases} \tag{3}$$

In the finite element method, over an element, the displacement is usually interpolated by

$$\begin{cases} u = \sum \phi_u^i u_i, \\ v = \sum \phi_v^i v_i, \end{cases} \tag{4}$$

where the subscript u or v is used to reflect the possible difference along the x and y directions for the shape functions.

Considering the versatility of the finite element method in approximating any possible variations, as a sufficient condition, on substituting Eq. (4) in Eq. (3), we stipulate that, on element level, the shape functions satisfy

$$\begin{cases} \nabla\cdot(\mu\nabla\phi_u^i) + \frac{\partial}{\partial x}\left[(\lambda+\mu)\frac{\partial\phi_u^i}{\partial x}\right] = 0, \\ \frac{\partial}{\partial y}\left[(\lambda+\mu)\frac{\partial\phi_u^i}{\partial x}\right] = 0 \end{cases} \tag{5a}$$

and

$$\begin{cases} \frac{\partial}{\partial x}\left[(\lambda+\mu)\frac{\partial\phi_v^i}{\partial y}\right] = 0, \\ \nabla\cdot(\mu\nabla\phi_v^i) + \frac{\partial}{\partial y}\left[(\lambda+\mu)\frac{\partial\phi_v^i}{\partial y}\right] = 0. \end{cases} \tag{5b}$$