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A compact difference scheme for a partial integro-differential equation with a weakly singular kernel ☆

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ABSTRACT

A compact difference scheme is presented for a partial integro-differential equation. The integral term is treated by means of the product trapezoidal method. The stability and L_2 convergence are proved by the energy method. The convergence order is $O(k^{3/2}+h^4)$. Two numerical examples are given to support the theoretical results.

Contents

1. Introduction

This paper is devoted to the study of a compact difference method for the partial integro-differential equation

$$u_t = \mu u_{xx} + \int_0^t (t-s)^{-1/2} u_{xx} ds, \quad 0 < x < 1, \ t \geqslant 0, \tag{1}$$

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with $\mu \geqslant 0$ and the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \geqslant 0, \tag{2}$$

and the initial condition

$$u(x,0) = u_0(x), \quad 0 \leqslant x \leqslant 1. \tag{3}$$

Equation similar to (1) can be found in the modeling of physical phenomena involving heat flow in materials with memory [1,2], phenomena associated with linear viscoelastic mechanics [3,4]. The integral term in (1) represents the viscosity part of the equation and $\mu \geqslant 0$ in (1) is a Newtonian contribution to the viscosity.

In the twentieth century decade, many considerable works on theoretical analysis [5–8,13–15] have been carried on. Yan and Fairweather [5] presented orthogonal spline collocation method for some partial integro-differential equations with smooth integral kernels. Xu [14,15] considered backward Euler method in time direction for a parabolic integro-differential equation and derived the stability and convergence properties of the time discretizations. Lopez-Marcos [10] studied the nonlinear partial integro-differential equation which is similar to problem (1)–(3), he used one order full discrete difference scheme and used a convolution quadrature to treat the integral term. A compact difference scheme is presented by Chen and Xu [8] for an evolution equation with a weakly singular kernel with the truncation error of order 3/2 in time and order 4 in space, the convergence and stability were obtained. The Crank–Nicolson scheme in time direction for solving problem (1)–(3) are provided by Tang [6], and the $O(k^{3/2} + h^2)$ order conditional convergence is proved. It is well known that the Crank–Nicolson scheme has $O(k^2)$ order accuracy, but due to the lack of smoothness of the integral kernel, the overall numerical procedure in [6] does not achieve second-order convergence. In this article, we give a compact difference scheme for problem (1)–(3) and proved that the compact difference scheme is stable and convergent in L_2 norm. The convergence order is $O(k^{3/2} + h^4)$.

Throughout the paper, we assume that u_0 in (3) is such that the problem (1)–(3) has a unique solution in $[0,1] \times [0,T]$. Furthermore, we suppose that u_{tt} and u_{ttxx} are continuous for $0 \leqslant x \leqslant 1$ and $0 \leqslant t \leqslant T$, and we assume that there exists a positive constant C_0 such that

$$|u_{tt}(x,t)| \leqslant C_0 t^{-1/2}, \quad |u_{ttt}(x,t)| \leqslant C_0 t^{-3/2}, \quad |u_{xxtt}(x,t)| \leqslant C_0 t^{-1/2}. \tag{4}$$

The outline of the paper is organized as follows: a compact finite difference scheme is introduced in Section 2. The analysis of stability and convergence of the scheme is given in Section 3. The numerical results are presented in Section 4. This paper ends with a conclusion.

2. The derivation of the compact difference scheme

We introduce a grid $w_h = \{x | x_j = jh, j = 0,1,\ldots,J\}$, $t_n = nk, n = 0,1,\ldots,N$ with $h = 1/J, k = 1/N$ and J, N are positive integers. Moreover, we let $t_{n+1/2} = (n+1/2)k, u_j^n = u(x_j,t_n), \ 0 \leqslant j \leqslant J, 0 \leqslant n \leqslant N$.

We first introduce the following product trapezoidal method to approximate $I(f,t) = \int_0^t (t-s)^{-1/2} f(s) ds$ which is introduced by Tang [6]:

$$I(f,t_n) = A_n f(t_0) + \sum_{p=0}^{n} \beta_p f(t_{n-p}) + O(k^{3/2}), \quad 1 \leqslant n \leqslant N, \tag{5}$$

where

$$A_n = 2\left[t_n^{1/2} - \frac{1}{k}\int_{t_n}^{t_n+1} \theta^{1/2} d\theta\right], \quad \beta_0 = \frac{2}{k}\int_0^{t_1} \theta^{1/2} d\theta + \frac{4\sqrt{k}}{3}\beta, \quad \beta_1 = \frac{2}{k}\left[\int_{t_1}^{t_2} \theta^{1/2} d\theta - \int_{t_0}^{t_1} \theta^{1/2} d\theta\right] - \frac{4\sqrt{k}}{3}\beta,$$
$$\beta_p = \frac{2}{k}\left[\int_{t_p}^{t_{p+1}} \theta^{1/2} d\theta - \int_{t_{p-1}}^{t_p} \theta^{1/2} d\theta\right], \quad p \geqslant 2. \tag{6}$$

where β is a nonnegative constant and is dependent of k and h. i.e., $\beta \geqslant 0$ and $\beta = O(1)$.

The following lemma will be used in the derivation of the compact difference scheme.

Lemma 2.1 ([9,11,12]). *Suppose $g(x) \in C^6[x_{i-1}, x_{i+1}]$. Then*

$$\frac{1}{12}[g''(x_{i-1}) + 10g''(x_i) + g''(x_{i+1})] - \frac{1}{h^2}[g(x_{i-1}) - 2g(x_i) + g(x_{i+1})] = \frac{h^4}{240} g^{(6)}(w_i), \quad w_i \in (x_{i-1}, x_{i+1}). \tag{7}$$

Lemma 2.2 [6]. *Let $I(f,t) = \int_0^t (t-s)^{-1/2} f(s) ds$, then*

$$I(f,t_{n+1/2}) = \frac{1}{2}[I(f,t_n) + I(f,t_{n+1})] + O(k^2 t_n^{-3/2}), \quad n \geqslant 1. \tag{8}$$