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Modelling torsion in an elastic cable in space

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Abstract

When an imperfectly flexible, elastic cable is suspended in a vertical plane under the sole influence of gravity and boundary conditions involving axial twist are then applied to the endpoints of the cable, its shape extends from the original vertical plane to a three-dimensional configuration in space. The aim of this paper is to develop a mathematical model (consisting of differential equations) for the configuration of and tension in such a twisted cable. The model is solved numerically for different boundary conditions. Although the final model delivers satisfactory results for small amounts of twist (inside the elastic deformation domain of the cable), realistic bounds for the amount of twist that may be applied (before plastic deformation of the cable causes model inaccuracies) are yet unknown. However, the effect of an increase in the amount of torsion applied at the endpoints of the cable is investigated numerically. The model seems capable of capturing first points of twisting bifurcation, when so much twist is applied at the endpoints that the cable jumps from un unstable equilibrium configuration to a more stable one.

1. Introduction

The problem of determining the shape of and tension in a perfectly flexible, inelastic cable suspended in a vertical plane under the sole influence of gravity is more than 400 years old and has an illustrious history. Leonardo da Vinci (1452–1519) was the first person known to express an

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interest in the *catenary*, while Galileo Galilei (1564–1642) seems to have been the first scientist to work on the problem. Galilei claimed that the shape is *parabolic*, but was proven incorrect by Ignatius Pardies, a member of the Jesuite Order in France. It was only in 1691, however, after a challenge by Jacob Bernoulli (1654–1705), that the true shape was found to be *hyperbolic cosine* by Johann Bernoulli (1667–1748), Christiaan Huygens (1629–1695) and Gottfried Wilhelm Leibniz (1646–1716) independently. This result was placed on a firmer mathematical foundation during 1715–1717 by Jacob Hermann (1668–1733) and Brook Taylor (1685–1731). There are also a number of famous variations to this problem, such as the *velaria* (the problem of determining the shape of a cable subjected to a normal load of constant magnitude, whose solution is a *circle*) and the problem of determining the shape of the supporting cables in a suspension bridge, which was found to be *parabolic* by Simon Stevin (1548–1620) and Isaac Beeckman (1588–1677).

There have been many generalisations of the original catenary problem since the seventeenth century, such as allowing for more general continuous loads on the cable than mere gravity, allowing for elasticity and bending stiffness in the cable, or accommodating a finite number of discrete point loads on the cable. Another interesting generalisation is that of determining the shape of an elastic, imperfectly flexible cable fixed in space, subjected at its endpoints to boundary conditions involving axial twist. A considerable amount of research has been done on the twist and torsion of prismic rods and beams. Jacob Bernoulli seems to have been the first person who published work on the bending of elastic rods. The foundations of rod statics are to be found in a 1691 paper by Bernoulli, as well as in a 1702 paper by Pierre Varignon. Euler [12] derived the general statical equations for a rod bent in its own plane in a paper published in 1771, while St. Venant [3] and Binet [4] produced, in 1843 and 1844 respectively, the first adequate mathematical description (consisting of six equations) of the concept of *strain* by introducing the notions of *twist* and *principal torsion-flexure axes*, albeit subject to various simplifying assumptions about the elastic properties of the rod. Other noteworthy attempts to formulate a theory of strain in rods include those of Clebsch [5] in 1862 and Kirchoff [16,17] in 1876, whose work is more general than that of St. Venant, but whose scope is still limited by the hypotheses that (i) the stress couple depends linearly upon curvature and twist, (ii) the axis of the rod is inextendable, (iii) there is no shear of the cross-section with respect to the rod axis, and (iv) there is no deformation within the cross-section. The 1893 paper by Love [19] made all the above mentioned work more accessible, while still being confined to small deformations and the notion of linear elasticity, as was the work of all his predecessors. The Cosserat pair [8,9] were the first authors to treat the twisting and bending of rods separately in 1907–1908, but their work is exceedingly difficult to follow, due to their use of Cartesian coordinates instead of suitable angular coordinates. Hay [13] employed tensor analysis in 1942 to describe the strain in a rod independently of the elastic hypotheses used.

The modern foundations of the theory of deformation of rods are due to Ericksen and Truesdell [11], whose 1958 paper contains a complete and general description of strain and *stress*, which is valid even for large deformations. Cohen [6] built on these foundations in 1966 by accommodating axial extension, bending, transverse shear and deformation of the rod's cross-section. Antman [1] obtained, in 1974, non-linear formulations for six deformation measures which are straight in their natural state, while Connor [7] utilised six measures of strain in rods which he called one-dimensional *deformation measures* in 1976 (two of these measures account for transverse shear deformations, two for changes in curvature, one for changes in twist and one for changes in rod length). An obvious, but valuable development, due to Koenig and Bolle [18] in 1993, was to obtain non-linear