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Some oscillation results for second-order nonlinear delay dynamic equations

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ABSTRACT

This paper is concerned with oscillatory behavior of a class of second-order delay dynamic equations on a time scale. Two new oscillation criteria are presented that improve some known results in the literature. The results obtained are sharp even for the second-order ordinary differential equations.

1. Introduction

In this paper, we are concerned with oscillation of a second-order nonlinear delay dynamic equation

$$\left(rx^{\Delta}\right)^{\Delta}(t) + q(t)f(x(\delta(t))) = 0, \tag{1.1}$$

where $t \in [t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$, and

(H_1) $r, q \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $r(t) > 0$, $q(t) > 0$;
(H_2) $\delta \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$, $\delta(t) \le t$, $\lim_{t\to\infty} \delta(t) = \infty$;
(H_3) $f \in C(\mathbb{R}, \mathbb{R})$ such that $yf(y) > 0$, $f(y)/y \ge K > 0$ for $y \ne 0$, where K is a constant.

Throughout this paper, we assume that solutions of (1.1) exist for any $t \in [t_0, \infty)_{\mathbb{T}}$. A solution x of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, we call it nonoscillatory. Dynamic equation (1.1) is said to be oscillatory if all its solutions oscillate.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above and is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$. On any time scale we define the forward and backward jump operators by $\sigma(t) := \inf\{s \in \mathbb{T} | s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} | s < t\}$, where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. Points that are right-scattered and left-scattered at the same time are called isolated. There are many time scales that consist of only isolated points; see, for example, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = q^{\mathbb{N}}$, and $\mathbb{T} = 2^{\mathbb{N}}$, etc. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t) := f(\sigma(t))$. Some concepts related to the notion of time scales; see Bohner and Peterson [1].

In recent years, there has been an increasing interest in obtaining sufficient conditions for oscillatory or nonoscillatory behavior of different classes of dynamic equations on time scales, we refer the reader to [1–24]. In what follows, we present

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some details that motivate the contents of this paper. Regarding oscillation of second-order delay equations, Zhang and Zhu [22] studied a dynamic equation

$$x^{\Delta\Delta}(t)+q(t)f(x(t-\delta))=0.$$

Agarwal et al. [2] considered an equation

$$x^{\Delta\Delta}(t)+q(t)x(\delta(t))=0.$$

Şahiner [18] investigated a nonlinear equation

$$x^{\Delta\Delta}(t)+q(t)f(x(\delta(t)))=0.$$

On the basis of condition

$$\int_{t_0}^{\infty}\frac{1}{r(t)}\int_{t_0}^{t}q(s)\Delta s\Delta t=\infty,$$

Erbe et al. [13] obtained a sufficient condition which ensures that the solution x of the delay dynamic equation (1.1) is either oscillatory or satisfies $\lim_{t\to\infty}x(t)=0$. Zhang [24] established some oscillation criteria for (1.1) in the case where

$$\int_{t_0}^{\infty}\frac{1}{r(t)}\int_{t_0}^{t}q(s)\int_{s}^{\infty}\frac{\Delta u}{r(u)}\Delta s\Delta t=\infty.$$

It is well known (see [20]) that the Euler differential equations

$$x''(t)+\frac{q_0}{t^2}x(t)=0,\quad q_0>0 \text{ is a constant} \tag{1.2}$$

and

$$(t^2x'(t))'+q_0x(t)=0,\quad q_0>0 \text{ is a constant} \tag{1.3}$$

are oscillatory if $q_0>1/4$. However, results obtained in [13,24] cannot give this conclusion for (1.3). The natural question now is: *Can one obtain new oscillation results for* (1.1), *which may cover* (1.2) *or* (1.3)? The aim of this paper is to give an affirmative answer to this question. The results reported improve those by [13,24].

2. Main results

In this section, we will establish two new Philos-type oscillation criteria for (1.1). For the sake of convenience, we use the notation

$$\mathbb{D}\equiv\{(t,s):t_0\le s\le t,\ t,s\in[t_0,\infty)_{\mathbb{T}}\}\quad\text{and}\quad\mathbb{D}_0\equiv\{(t,s):t_0\le s<t,\ t,s\in[t_0,\infty)_{\mathbb{T}}\}.$$

All functional inequalities considered in this section are assumed to hold eventually, that is, they are satisfied for all t large enough.

Theorem 2.1. *Assume* (H_1)–(H_3) *and*

$$\int_{t_0}^{\infty}\frac{\Delta t}{r(t)}=\infty. \tag{2.1}$$

Suppose further that there exist two functions $\eta, a\in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that* $\eta(t)>0$, $a(t)\ge 0$, *and there exists a function* $H\in C_{rd}(\mathbb{D},\mathbb{R})$ *such that*

$$H(t,t)=0,\quad t\ge t_0;\qquad H(t,s)>0,\quad t>s\ge t_0, \tag{2.2}$$

and H *has a nonpositive rd-continuous* Δ*-partial derivative* $H^{\Delta_s}(t,s)$ *on* $\mathbb{D}_0$ *with respect to the second variable and satisfies*

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),t_2)}\int_{t_2}^{t}\left[H(\sigma(t),\sigma(s))Q(s)-\frac{(H(\sigma(t),\sigma(s))B(s)+H^{\Delta_s}(\sigma(t),s))^2}{4H(\sigma(t),\sigma(s))A(s)}\right]\Delta s=\infty \tag{2.3}$$

for all sufficiently large t_1 *and for some* $t_2\ge t_1$, *where*

$$Q(s):=\eta^{\sigma}(s)\left[Kq(s)\frac{\int_{t_1}^{\delta(s)}\frac{\Delta v}{r(v)}}{\int_{t_1}^{\sigma(s)}\frac{\Delta v}{r(v)}}+r(s)a^2(s)\frac{\int_{t_1}^{s}\frac{\Delta v}{r(v)}}{\int_{t_1}^{\sigma(s)}\frac{\Delta v}{r(v)}}-(r(s)a(s))^{\Delta}\right],$$

$$A(s):=\frac{\eta^{\sigma}(s)}{r(s)\eta^2(s)}\frac{\int_{t_1}^{s}\frac{\Delta v}{r(v)}}{\int_{t_1}^{\sigma(s)}\frac{\Delta v}{r(v)}},\qquad B(s):=\frac{\eta^{\Delta}(s)}{\eta(s)}+\frac{2\eta^{\sigma}(s)a(s)}{\eta(s)}\frac{\int_{t_1}^{s}\frac{\Delta v}{r(v)}}{\int_{t_1}^{\sigma(s)}\frac{\Delta v}{r(v)}}.$$

Then (1.1) *is oscillatory.*

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