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Survival to extinction in a slowly varying harvested logistic population model

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ABSTRACT

This work considers a harvested logistic population for which birth rate, carrying capacity and harvesting rate all vary slowly with time. Asymptotic results from earlier work, obtained using a multiscaling technique, are combined to construct approximate expressions for the evolving population for the situation where the population initially survives to a slowly varying limiting state, but then, due to increasing harvesting, is reduced to extinction in finite time. These results are shown to give very good agreement with those obtained from numerical computation.

1. Introduction

In earlier work (see Idlango et al. [1]), we considered the evolution of a harvested logistic population represented by the initial value problem

$$\frac{\mathrm{d}p(t,\epsilon)}{\mathrm{d}t} = r(\epsilon t)\, p(t,\epsilon)\left(1 - \frac{p(t,\epsilon)}{k(\epsilon t)}\right) - \sigma\, h(\epsilon t), \quad p(0,\epsilon) = \mu. \tag{1}$$

In (1), all quantities are dimensionless, with p representing the evolving population at times $t \geq 0$, while r, k and h are growth rate, carrying capacity and harvesting rate, respectively. Both σ and ϵ are positive constants, with σ denoting the weight or magnitude of the harvesting term, while ϵ represents the ratio of the timescales of variation of r, k and h to the intrinsic timescale of variation of p. Implicit in this is the assumption that all of r, k and h vary on the same timescale (which reduces to the intrinsic scale for p when $\epsilon = 1$). The initial population is given by positive μ.

The particular notation of (1) emphasizes the fact that in what follows (and as was the case in [1]), we focus attention on the dependence on ϵ of p, with dependence on σ and μ being suppressed.

At this point, we recall that, for arbitrarily varying r, k and h, the problem (1) is virtually impossible to solve exactly, and so numerical solutions *must* be used. However, here as in [1], we direct attention to the situation where ϵ is small (and positive). This characterizes slow variation in r, k and h (relative to that of p); and in this circumstance, as we showed in [1], a multiscaling analysis of the problem (1) leads to approximate solutions of (1) that (formally, at least) represent $p(t,\epsilon)$ on all $t \geq 0$, and which take the form of a two-term approximation to $p(t,\epsilon)$, namely

$$p(t,\epsilon) = p_0(t,\epsilon) + \epsilon\, p_1(t,\epsilon) + O(\epsilon^2), \tag{2}$$

valid on $t \geq 0$ for small $\epsilon > 0$.

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The detailed forms of p_0 and p_1 were shown to depend on the *indicator* $\delta(t_1)$, given by

$$\delta(t_1) = 1 - \frac{4\,\sigma\, h(t_1)}{r(t_1)\, k(t_1)}, \tag{3}$$

while each of p_0 and p_1 was shown to depend on two timescales: t_0, a 'normal' time, and t_1, a 'slow' time defined by

$$t_0 = \frac{1}{\epsilon}\int_0^{t_1} r(s)\,\chi(s)\,\mathrm{d}s, \quad t_1 = \epsilon\, t, \tag{4}$$

respectively, where $\chi(t_1) = \sqrt{|\delta(t_1)|}$.

Two situations are significant—*subcritical harvesting*, when $\delta(t_1) > 0$ on $t_1 \geq 0\,(t \geq 0)$, and *supercritical harvesting*, when $\delta(t_1) < 0$ on $t_1 \geq 0\,(t \geq 0)$.

For subcritical harvesting, the behaviour of $p_0,\ p_1$ and the approximation (2) vary, depending on the value of μ:

- When

$$\mu > \frac{1}{2}k(0)\,(1-\chi(0)), \tag{5}$$

both p_0 and p_1 show an initial 'transient' region, in which t_0 variation dominates; but as t_0 increases, the expansion (2) tends to the slowly varying limiting state

$$\frac{1}{2}k(t_1)\,\{1+\chi(t_1)\} - \epsilon\,\left\{\frac{(k(t_1)\,\chi(t_1))' + k'(t_1)}{2\,r(t_1)\chi(t_1)}\right\} + O(\epsilon^2). \tag{6}$$

This is *subcritical harvesting with survival*.

- When

$$0 < \mu < \frac{1}{2}k(0)\,(1-\chi(0)), \tag{7}$$

to the orders considered, the approximation (2) reaches zero in a finite time.

This is *subcritical harvesting with extinction*.

For supercritical harvesting, regardless of the value of μ, both p_0 and p_1 vanish for some finite t_1 (or t) value, so, to the level of approximation considered in [1], the population reaches zero in finite time.

This is *supercritical harvesting with extinction*.

Note that in the above, and in the discussion to follow, we interpret strict inequalities to be independent of ϵ as $\epsilon \to 0$. Thus, '$\delta(t_1) > 0$' is to be interpreted as '$\delta(t_1) \geq \delta_0 > 0$ where δ_0 is independent of ϵ', with analogous interpretations of other inequalities.

As we have noted in [1], the validity of the expansion (2) on $t \geq 0$ is highly dependent on the properties of $\chi(t_1)$ and hence $\delta(t_1)$ on $t_1 \geq 0$. In each case, $p_1(t,\epsilon)$ included a term involving $\chi(t_1)$ in the denominator, with this term (and hence the expansion (2)) becoming undefined at any point $\bar{t}_1$ where $\chi(\bar{t}_1) = \delta(\bar{t}_1) = 0$. More generally, at any point t_1 where $\chi(t_1) = O(\epsilon)$, the expansion (2) becomes *disordered*, with the second term becoming comparable with the first, so this expansion again fails there. This last occurs in a neighbourhood of points $\bar{t}_1$ where $\delta(\bar{t}_1) = 0$. In particular, this will arise at points where $\delta(t_1)$ changes sign—i.e., where the harvesting goes from subcritical to supercritical, or vice versa. We will term such points *transition points*.

In what follows, we will focus on the situation where the population changes from subcritical harvesting with survival to a supercritical with extinction state. This occurs when the harvesting is subcritical, but increases over time to become supercritical leading the population to a decline and eventual extinction.

Thus, in what follows we will assume that $\delta(t_1)$ changes from positive to negative at a single point $t_1 = \bar{t}_1$ on $t_1 > 0$; and that this zero of $\delta(t_1)$ is simple. This last notion leads us to assume that

$$\delta(\bar{t}_1) = 0 \quad \text{and} \quad \delta'(\bar{t}_1) < 0. \tag{8}$$

Recalling the inequalities convention above, we define three regions of the t_1-axis:

R_p (positive region), given by $0 \leq t_1 < \bar{t}_1$, on which $\delta(t_1) > 0$;
R_t (transition region), a neighbourhood of $\bar{t}_1$, in which $\delta(t_1)$ changes sign;
R_n (negative region), given by $t_1 > \bar{t}_1$, on which $\delta(t_1) < 0$.

In the following sections, we will use our earlier results [1] to construct approximations to the solution of (1) on regions R_p and R_n, while a separate (local) analysis will construct such an approximation on region R_t. This last notion will be linked to the others using a matching technique, and the approximations on all three regions combined to form a uniform approximation over the whole of $t_1 \geq 0\,(t \geq 0)$, valid as $\epsilon \to 0$. This approach parallels the analysis of Shepherd et al. [2] and Grozdanovski et al. [3] for transitions in the logistic and power law logistic models, and is based on the extensive study by Haberman [4].