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Sharp criteria of global existence and blow-up for a type of nonlinear parabolic equations ☆

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ABSTRACT

We study a type of nonlinear parabolic equations. In terms of the variational characterization of the corresponding nonlinear elliptic equations and the invariant flow arguments, we establish the sharp criteria for global existence and blow-up. Furthermore, we also get the instability of the steady states and the global existence with small initial data.

1. Introduction

In this paper, we study the Cauchy problem for a type of nonlinear parabolic equation in $H^1(\mathbb{R}^N)$ with $N \geqslant 2$,

$$u_t - \Delta u + u = |u|^{q-1}u, \quad t > 0,\ x \in \mathbb{R}^N, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_0(x) \in H^1(\mathbb{R}^N), \tag{1.2}$$

where $u = u(t,x) : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$, and $1 < q < \infty$ for $N = 2$, and $1 < q < \frac{N+2}{N-2}$ for $N \geqslant 3$. We also denote $\frac{N+2}{(N-2)^+} := \begin{cases} \infty, & \text{when } N = 2, \\ \dfrac{N+2}{N-2}, & \text{when } N \geqslant 3. \end{cases}$

Wang and Ding [1] studied the positive solution for the Eq. (1.1) in $L^p(\mathbb{R}^N)$. They showed a sharp threshold for the global existence and blow-up of the Eq. (1.1) in terms of the positive solution of the corresponding nonlinear elliptic equation by applying the comparing principle.

We consider the nonlinear elliptic equation

$$-\Delta v + v = |v|^{q-1}v, \quad v \in H^1(\mathbb{R}^N), \tag{1.3}$$

where $1 < q < \frac{N+2}{(N-2)^+}$. It is well known that for (1.3), there exists a unique positive symmetric solution $Q(x)$. Wang and Ding's results [1] read as follows:

(I) If $0 \leqslant u_0(x) \leqslant Q(x)$, there exists a L^p global solution of the Eq. (1.1) with $u_0 \in L^P(\mathbb{R}^N)(1 \leqslant p \leqslant \infty)$;
(II) If $u_0(x) \geqslant Q(x)$, and $u_0(x) \not\equiv Q(x)$, the $L^p(2 < p \leqslant \infty)$ solutions blow up in a finite time.

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This result is elegant but it is difficult to verify an initial datum $u_0(x) \geqslant Q(x)$ or $u_0(x) \leqslant Q(x)$, for all $x \in \mathbb{R}^N$. In other words, this is a local result with point to point.

In this paper, we try to give an integral result for the sharp criteria of the global existence and blow-up of the Eq. (1.1). We exploit the variational characterization of the nonlinear elliptic equation (1.3) by constructing the proper variational problem. Then we establish the invariant flow of the Cauchy problem (1.1)–(1.2). Thus we can get a sharp criteria for the global existence and blow-up of the Eq. (1.1). Furthermore, we also get the instability of the steady states and the global existence with small initial data. The method and idea are essentially different from [1].

In some surprise, the results in this paper imply that the nonlinear parabolic equation (1.1) has similar properties as the nonlinear Klein–Gordon equation, which is a type of the nonlinear wave equation (see [2,3]). In fact the argument used in the following is inspired by the related study of the nonlinear wave equations, see [4,2,5,3,6].

2. Preliminaries

We consider the Cauchy problem (1.1)–(1.2). From [7–9], the following local well-posedness is true.

Proposition 2.1. *If $u_0(x) \in H^1(\mathbb{R}^N)$, then there exists a unique solution $u(t,x)$ of the Cauchy problem (1.1)–(1.2) on the maximal time $[0,T)$ such that $u(t,x) \in C([0,T], H^1(\mathbb{R}^N))$ and either $T = \infty$ (global existence), or $T < \infty$ and*

$$\lim_{t\to T} \|u(t,x)\|_{H^1(\mathbb{R}^N)} = \infty \quad \textit{(blowing up)}.$$

Now we define two functionals for $v \in H^1(\mathbb{R}^N)$ and $1 < q < \frac{N+2}{(N-2)^+}$,

$$E(v) := \frac{1}{2}\int |\nabla v|^2 dx + \frac{1}{2}\int |v|^2 dx - \frac{1}{q+1}\int |v|^{q+1} dx, \tag{2.1}$$

and

$$K(v) := \int |\nabla v|^2 dx + \int |v|^2 dx - \int |v|^{q+1} dx. \tag{2.2}$$

From Sobolev's embedding theorem, the above definitions are definite. In the following, we denote $\int_{\mathbb{R}^N} \cdot dx$ by $\int \cdot dx$ for simplicity.

We further define a set

$$M := \{v \in H^1(\mathbb{R}^N) \setminus \{0\} \mid K(v) = 0\}. \tag{2.3}$$

Then we define a constrained variational problem as follows.

$$d := \inf_{v\in M} E(v). \tag{2.4}$$

From [10], one has that

Proposition 2.2. *Let $Q(x)$ be the unique positive symmetric solution of the Eq. (1.3) Then*

$$d = E(Q) = \min_{v\in M} E(v). \tag{2.5}$$

We also have the following proposition.

Proposition 2.3. *Let $1 < q < \frac{N+2}{(N-2)^+}$ and $u_0(x) \in H^1(\mathbb{R}^N)$. For the solution $u(t,x)$ of the Cauchy problem (1.1)–(1.2) on the maximal time $[0,T)$, one has that*

$$\frac{dE(u(t,\cdot))}{dt} = -\int |u_t(t,\cdot)|^2 dx, \quad t \in [0,T) \tag{2.6}$$

and then

$$E(u(t,\cdot)) \leqslant E(u_0), \quad t \in [0,T). \tag{2.7}$$

3. Sharp criteria

In this section, we imply the main results of this paper. Firstly we establish the invariant flow of the Cauchy problem (1.1)–(1.2) by Propositions 2.1–2.3.

Lemma 3.1. *Let*

$$S_1 := \{v \in H^1(\mathbb{R}^N), K(v) > 0, E(v) < E(Q)\}, \tag{3.1}$$

$$S_2 := \{v \in H^1(\mathbb{R}^N), K(v) < 0, E(v) < E(Q)\}. \tag{3.2}$$