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An iterative splitting approach for linear integro-differential equations

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ABSTRACT

The motivation for this paper is to solve a model based on the dynamics of electrons in a plasma using a simplified Boltzmann equation. Such problems have arisen in active plasma resonance spectroscopy, which is used for plasma diagnostic techniques; see Braithwaite and Franklin (2009) [1]. We propose a modified iterative splitting approach to solve the Boltzmann equations as a system of integro-differential equations. To enable solution by fast and iterative computations, we first transform the integro-differential equations into second order differential equations. Second, we split each second order differential equations into two first order differential equations via a splitting approach. We carry out an error analysis of the higher order iterative approach. Numerical experiments with a simplified Boltzmann equation will be discussed, along with the benefits of computing with this splitting approach.

1. Introduction

We motivate our study by simulating an active plasma resonance spectroscopy, something which is well established among plasma diagnostic techniques. To study such a model with fast solver schemes, we concentrate on an abstract kinetic model which describes the dynamics of the electrons in a plasma by applying the Boltzmann equation. Based on the modelling in [1], we use a simplified Boltzmann equation in the form of an integro-differential equation and we assume that the velocities are nearly constant:

$$\frac{\partial u(x,t)}{\partial t} = -v \cdot \nabla u(x,t) + \int_0^t \kappa(t') u(x,t')\, dt', \qquad u(x,0) = u_0(x), \quad (x,t) \in \Omega \times [0,T], \tag{1}$$

where the boundary conditions are assumed to be periodic and the kernel operator κ is linear and bounded. We assume that $\kappa(t)$ can be approximated and separated as $\kappa(t) \approx (\kappa_0 - t_0\kappa_1) + t\kappa_1$, where $\kappa_0 = \kappa(t_0)$, $\kappa_1 = \frac{\partial \kappa}{\partial t}|_{t_0}$ and t_0 is the previous time–point used for the linearization. For such a model, we are interested in the influence of the transport and scattering processes.

Historically, numerical methods for solving integro-differential equations were given explicitly and the integral operator was solved via an integration by parts; see [2,3].

In recent papers, the authors have applied splitting schemes, see [4,5], to decouple the nonlinear differential part from the linear integral part of the equation. The idea is to solve each separate system of equations independently, more quickly, and thus save computational time [6]. The spatially discretized equations are approximately solved by fast matrix exponential solvers and accelerated splitting schemes; see [7].

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Splitting schemes are important if they can reduce the computational cost of solving the full system by decomposing it into stable and simpler numerical schemes; see [8]. Furthermore, the splitting of the full matrix given by the spatial discretization into two independent symmetric positive definite sub-schemes reduces the computational time and circumvents a delicate saddle point problem; see [9].

To overcome the time-consuming integral part, Ramos [10] discussed a decomposition with respect to iterative and non-iterative methods. While iterative methods are based on fixed point schemes, see [11], the non-iterative methods are based on finding a solution in the form of a series of the integral part; see [10]. For the linear and nonlinear integral parts, the application of iterative methods gets more important, to approximate the operator equations and accelerate the solver process [12].

We concentrate on the iterative methods and reformulate the integro-differential equation into a system of first order differential equations; see [3,13]. Based on this idea, we can obtain the benefits of fast iterative splitting schemes, which are constructed for first order systems, see [14,15], and circumvent the expensive computations for the full discretized system or the matrix exponential system; see [16].

This paper is organized as follows. Section 2 derives the mathematical equations of the model. The splitting approach and the error analysis are presented in Section 3. The numerical experiments are reported in Section 4 and we summarize our results in Section 5.

2. Model equations based on linear integro-differential equations

We assume that the model Eq. (1) is semidiscretized with finite differences or finite element methods and can be reformulated as an ordinary integro-differential equations. In the following, we treat an abstract homogeneous Cauchy problem in a Banach space $\mathbf{X} \subseteq \mathbb{R}^n$:

$$\frac{d\mathbf{u}(t)}{dt} = L\mathbf{u} = A\mathbf{u}(t) + B\int_0^t \mathbf{u}(s)\, ds + \mathbf{f}(t), \quad \mathbf{u}(0) = \mathbf{u}_0, \tag{2}$$

where $A, B \in \mathbf{X} \times \mathbf{X}$ are bounded operators, $\|\cdot\|$ is the corresponding norm in $\mathbf{X}$, and $\|\cdot\|_{L(\mathbf{X})}$ is the induced operator norm. Based on the kernel-function, see (1), in the abstract formulation, we write the operator B outside of the integral. The time dependent parts are included on the right hand side $\mathbf{f} \in \mathbf{X}$, which is a perturbation; see [11]. A corresponds to the discretized transport operator in Eq. (1), e.g., with a finite differences scheme. B is the time-discretized and approximated kernel operator in Eq. (1). Based on the semidiscretization, the boundary conditions are also embedded in the matrices A and B. For the splitting approach, we modify these to be second order differential equations and make the following assumptions.

Assumption 2.1. 1. The function $\mathbf{f}$ is given by $\mathbf{f} = B\mathbf{U}(0)$, and we assume $\kappa_1 \approx 0$. Furthermore, the primitive is $\int_0^t \mathbf{u}(s)\, ds = \mathbf{U}(t) - \mathbf{U}(0)$. Without loss of generality, we may assume in the following that $\mathbf{U}(0) = \mathbf{0}$.
2. The initial condition of the first derivative is given as $\frac{d\mathbf{u}}{dt}|_{t=0} = A\mathbf{u}(0)$.

Corollary 2.1. *The integro-differential equation (2) can be rewritten, thanks to Assumptions 2.1, as a second order differential equation:*

$$\frac{d^2\mathbf{u}(t)}{d^2t} = A\frac{d\mathbf{u}(t)}{dt} + B\mathbf{u}(t), \quad \mathbf{u}(0) = \mathbf{u}_0, \qquad \frac{d\mathbf{u}}{dt}|_{t=0} = A\mathbf{u}_0. \tag{3}$$

Here, the analytical solution is

$$\mathbf{u}(t) = \frac{1}{2}\left(\exp\left(\left(\frac{A}{2} + \sqrt{AA^t/4 + B}\right)t\right) + \exp\left(\left(\frac{A}{2} - \sqrt{AA^t/4 + B}\right)t\right)\right)\mathbf{u}_0. \tag{4}$$

If we assume A and $\sqrt{AA^t/2 + B}$ commute, the analytical solution reduces to

$$\mathbf{u}(t) = \exp\left(\frac{A}{2}t\right)\cosh(\sqrt{AA^t/4 + B}t)\mathbf{u}_0. \tag{5}$$

Proof. The analytical solution of the second order differential equation (3) is found by using the characteristic polynomial:

$$\lambda^2 - A\lambda - B = 0, \tag{6}$$

where we decompose this into simpler polynomials and obtain the roots:

$$\left(\lambda - \left(\frac{A}{2} + \sqrt{AA^t/4 + B}\right)\right)\left(\lambda - \left(\frac{A}{2} - \sqrt{AA^t/4 + B}\right)\right) = 0, \qquad \lambda_{1,2} = \frac{A}{2} \pm \sqrt{AA^t/4 + B}. \tag{7}$$

We apply the solution to our integro-differential equation (2) and using Assumptions 2.1, the equation is fulfilled. □

The next simplification is carried out to rewrite the integral–differential equation as two first order differential equations. Such a reduction allows us to apply fast and accurate higher order iterative splitting methods.