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On the boundedness of a class of nonlinear dynamic equations of second order☆

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ABSTRACT

In this paper, a generalized nonlinear dynamic integral inequality on time scales is established and then is used to study the boundedness of a class of nonlinear second-order dynamic equations on time scales. These theorems contain as special cases results for second-order differential equations, difference equations and q-difference equations.

1. Introduction

To unify and extend continuous and discrete analyses, the theory of time scales was introduced by Hilger [1] in his Ph.D. Thesis in 1988. Since then, the theory has been evolving, and it has been applied to various fields of mathematics; for example, see [2,3] and the references therein. It is well known that Gronwall-type integral inequalities and their discrete analogues play a dominant role in the study of quantitative properties of solutions of differential, integral and difference equations. During the last few years, some Gronwall-type integral inequalities on time scales and their applications have been investigated by many authors. For example, we refer readers to [4–10]. In this paper, motivated by the paper [5], we establish a new nonlinear Gronwall–Bellman type dynamic inequality on time scales, and then using the dynamic inequality we obtain the bounds on the solutions of a class of nonlinear dynamic equations of second order on time scales, which has generalized the main result of [5]. For all the detailed definitions, notation and theorems on time scales, we refer the readers to the excellent monographs [2,3] and references given therein. We also present some preliminary results that are needed in the remainder of this paper as useful lemmas for the discussion of our proof.

In what follows, R denotes the set of real numbers, $\mathrm{R}_+ = [0, +\infty)$; $C(M, S)$ denotes the class of all continuous functions defined on set M with range in set S, T is an arbitrary time scale, and C_{rd} denotes the set of rd-continuous functions. Throughout this paper, we always assume that $t_0 \in \mathrm{T}$, $\mathrm{T}_0 = [t_0, +\infty) \cap \mathrm{T}$.

2. A nonlinear Bihari-type dynamic inequality and some lemmas

Lemma 2.1 ([9]). *Suppose $u(t), a(t) \in C_{rd}(\mathrm{T}_0, \mathrm{R}_+)$, and a is nondecreasing, $f(t, s), f_t^{\Delta}(t, s) \in C_{rd}(\mathrm{T}_0 \times \mathrm{T}_0, \mathrm{R}_+)$, $\omega \in C(\mathrm{R}_+, \mathrm{R}_+)$ is nondecreasing. If for $t \in \mathrm{T}_0$, $u(t)$ satisfies the following inequality*

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$$u(t) \le a(t) + \int_{t_0}^{t} f(t,s)\omega(u(s))\Delta s, \quad t \in \mathrm{T}_0 \tag{2.1}$$

then

$$u(t) \le G^{-1}\left[G(a(t)) + \int_{t_0}^{t} f(t,s)\Delta s\right], \quad t \in \mathrm{T}_0 \tag{2.2}$$

where G is an increasing bijective function, and

$$[G(z(t))]^{\Delta} = \frac{z^{\Delta}(t)}{\omega(z(t))}. \tag{2.3}$$

Definition (*[11]*)**.** A function $g \in C(\mathrm{R}_+, \mathrm{R}_+)$ is said to belong to the class of $\mathcal{F}$ if

(i) g is nondecreasing, and
(ii) $\frac{g(u)}{v} \le g\left(\frac{u}{v}\right)$ for $u \ge 0$, $v \ge 1$.

It is easy to see that $g(u) \in \mathcal{F}$ implies $\int_1^{+\infty} \frac{1}{g(s)}\Delta s = +\infty$.

Lemma 2.2. *Suppose* (i) $u(t)$ *and* $a(t) \in C_{rd}(\mathrm{T}_0, \mathrm{R}_+)$, $a(t) \ge 1$ *is nondecreasing on* T_0*;* (ii) $f_i(t,s), f_i^{\Delta}(t,s) \in C_{rd}(\mathrm{T}_0 \times \mathrm{T}_0, \mathrm{R}_+)$*;* (iii) $h_i \in \mathcal{F}\,(i = 1, 2, \ldots, m)$*. If for* $t \in \mathrm{T}_0$, $u(t)$ *satisfies the following inequality*

$$u(t) \le a(t) + \sum_{i=1}^{m} \int_{t_0}^{t} f_i(t,s) h_i[u(s)]\Delta s, \tag{2.4}$$

then

$$u(t) \le a(t) \prod_{i=1}^{m} L_i(t), \quad t \in \mathrm{T}_0 \tag{2.5}$$

where

$$\begin{cases} L_i(t) = G_i^{-1}\left\{G_i(1) + \int_{t_0}^{t} f_i(t,s)\left(\prod_{k=1}^{i-1} L_k(s)\right)\Delta s\right\}, \\ G_i^{\Delta}(z(t)) = \dfrac{z^{\Delta}(t)}{h_i(z(t))}, \quad t \in \mathrm{T}_0 \\ \prod_{k=1}^{0} L_k(t) = 1, \quad i = 1, 2, \ldots, m. \end{cases} \tag{2.6}$$

Proof. We shall use induction to prove the conclusion. Let $m = 1$, (2.4) becomes

$$u(t) \le a(t) + \int_{t_0}^{t} f_1(t,s) h_1[u(s)]\Delta s, \quad t \in \mathrm{T}_0.$$

For arbitrary $T \in \mathrm{T}_0$, when $t \in [t_0, T]$, from the last inequality we have

$$\frac{u(t)}{a(T)} \le 1 + \int_{t_0}^{t} f(T,s) h_1\left[\frac{u(s)}{a(T)}\right]\Delta s$$

for $t \in [t_0, T]$. By Lemma 2.1 we have

$$\frac{u(t)}{a(T)} \le G^{-1}\left[G(1) + \int_{t_0}^{t} f(T,s)\Delta s\right],$$

i.e.,

$$u(t) \le a(T) G^{-1}\left[G(1) + \int_{t_0}^{t} f(T,s)\Delta s\right], \quad t \in [t_0, T].$$

Since $T \in \mathrm{T}_0$ is arbitrary, we have proved the validity of (2.5) when $m = 1$. Now suppose that (2.5) is true when $m = 1$, i.e.,

$$u(t) \le a(t) + \sum_{i=1}^{k} \int_{t_0}^{t} f_i(t,s) h_i[u(s)]\Delta s \quad t \in \mathrm{T}_0$$