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Explicit formulas for the exponentials of some special matrices

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ABSTRACT

The matrix exponential plays a very important role in many fields of mathematics and physics. It can be computed by many methods. This work is devoted to the study of some explicit formulas for computing $e^{\mathbf{A}}$, where $\mathbf{A}$ is a special square matrix. The main results are based on the convergent power series of $e^{\mathbf{A}}$. Examples and applications are given.

1. Introduction

Consider a linear first-order constant coefficient ordinary differential equation encountered in the study of dynamical systems and linear systems:

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}x(t), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $x(t)$ and x_0 are n-vectors, and $\mathbf{A}$ is an $n \times n$ matrix of complex constants.

It is well known that the solution to this equation is given by

$$x(t) = e^{\mathbf{A}t}x_0, \tag{2}$$

where $e^{\mathbf{A}t}$ denotes the exponential of the matrix $\mathbf{A}t$ and can be defined by the convergent power series

$$e^{\mathbf{A}t} = \sum_{i=0}^{\infty} \frac{(\mathbf{A}t)^i}{i!}. \tag{3}$$

It is therefore important to have an accurate numerical method for computing the matrix exponential function. Many methods for computing $e^{\mathbf{A}}$ were widely studied in [1] and an update [2]. Some improved approaches have been proposed in [3–5]. But in fact, most of the methods presented in the literature for computing $e^{\mathbf{A}}$ always have drawbacks as regards computational stability and efficiency.

From the theoretical and practical points of view, it is clear that explicit formulas for the matrix exponential are effective because truncation errors can be avoided. Therefore, more explicit formulas have been developed for the matrix exponential by many authors. For example, Apostol [6], Thompson [7], and Politi [8] gave explicit formulas for some special cases. Bensaoud and Mouline [9], and Taher and Rachidi [10] gave explicit formulas for the general cases. Bernstein and So [11] gave explicit formulas for $n = 2$ and for some special cases when $n > 2$. Cheng and Yau [12] gave explicit formulas for $n = 3, 4$ and for the general cases.

In this work we are primarily concerned with $n \times n$ matrices that satisfy some special polynomials. The explicit formulas, which yield some well-known formulas like those in [11,12] as special cases, are derived for the matrix exponential. The

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scope of our formulas is more extensive than those of the ones that we found in [11]. Compared with the methods in [9,10], our method for computing $e^{\mathbf{A}}$ is simpler and more straightforward even though it is only valid for some special cases. Finally, examples and applications are given to illustrate our results.

2. Main results

Denote the set of nonnegative integers by $\mathbf{N}_0$, the set of complex numbers by $\mathbf{C}$, and the set of all $n \times n$ complex matrices by $\mathbf{C}^{n\times n}$. The symbols $\mathbf{0}_n$ and $\mathbf{I}_n$ will be used to denote the n-by-n zero matrix and the n-by-n identity matrix, respectively.

Bernstein and So [11] gave explicit formulas for $\mathbf{A}^2 = \mathbf{A}$, $\mathbf{A}^2 = \rho\mathbf{I}_n$ and $\mathbf{A}^3 = \rho\mathbf{A}$, $\rho \in \mathbf{C}$. Now we generalize the results to the general cases.

Theorem 1. *Let* $\mathbf{A} \in \mathbf{C}^{n\times n}$, *where* $\mathbf{A}^{k+1} = \rho\mathbf{A}^k,\ \rho \in \mathbf{C},\ k \in \mathbf{N}_0$.

(i) *If* $\rho = 0$, *then* $e^{\mathbf{A}} = \sum_{i=0}^{k} \frac{\mathbf{A}^i}{i!}$.
(ii) *If* $\rho \neq 0$, *then*

$$e^{\mathbf{A}} = \sum_{i=0}^{k-1} \frac{\mathbf{A}^i}{i!} + \frac{e^{\rho} - \sum\limits_{i=0}^{k-1} \frac{\rho^i}{i!}}{\rho^k}\mathbf{A}^k. \tag{4}$$

Proof. (i) For $\rho = 0$, it is immediately obtained that

$$e^{\mathbf{A}} = \sum_{i=0}^{\infty} \frac{\mathbf{A}^i}{i!} = \sum_{i=0}^{k} \frac{\mathbf{A}^i}{i!}$$

since $\mathbf{A}^{k+1} = \mathbf{0}_n,\ k \in \mathbf{N}_0$.

(ii) For $\mathbf{A}^{k+1} = \rho\mathbf{A}^k,\ \rho \neq 0$, we have

$$\begin{aligned} e^{\mathbf{A}} &= \sum_{i=0}^{\infty} \frac{\mathbf{A}^i}{i!} \\ &= \sum_{i=0}^{k-1} \frac{\mathbf{A}^i}{i!} + \left(\sum_{i=k}^{\infty} \frac{\rho^{i-k}}{i!}\right)\mathbf{A}^k \\ &= \sum_{i=0}^{k-1} \frac{\mathbf{A}^i}{i!} + \frac{e^{\rho} - \sum\limits_{i=0}^{k-1} \frac{\rho^i}{i!}}{\rho^k}\mathbf{A}^k. \quad \square \end{aligned}$$

Theorem 2. *Let* $\mathbf{A} \in \mathbf{C}^{n\times n}$, *where* $\mathbf{A}^{k+2} = \rho^2\mathbf{A}^k$, $\rho \in \mathbf{C}$, $k \in \mathbf{N}_0$.

(i) *If* $\rho = 0$, *then* $e^{\mathbf{A}} = \sum_{i=0}^{k+1} \frac{\mathbf{A}^i}{i!}$.
(ii) *If* $\rho \neq 0$, $k = 2l$ $(l \in \mathbf{N}_0)$, *then*

$$e^{\mathbf{A}} = \sum_{i=0}^{k-1} \frac{\mathbf{A}^i}{i!} + \frac{\cosh(\rho) - \sum\limits_{m=0}^{k/2-1} \frac{\rho^{2m}}{(2m)!}}{\rho^k}\mathbf{A}^k + \frac{\sinh(\rho) - \sum\limits_{m=0}^{k/2-1} \frac{\rho^{2m+1}}{(2m+1)!}}{\rho^{k+1}}\mathbf{A}^{k+1}. \tag{5}$$

(iii) *If* $\rho \neq 0$, $k = 2l + 1$ $(l \in \mathbf{N}_0)$, *then*

$$e^{\mathbf{A}} = \sum_{i=0}^{k-1} \frac{\mathbf{A}^i}{i!} + \frac{\sinh(\rho) - \sum\limits_{m=0}^{(k-1)/2-1} \frac{\rho^{2m+1}}{(2m+1)!}}{\rho^k}\mathbf{A}^k + \frac{\cosh(\rho) - \sum\limits_{m=0}^{(k-1)/2} \frac{\rho^{2m}}{(2m)!}}{\rho^{k+1}}\mathbf{A}^{k+1}. \tag{6}$$

Proof. (i) Obvious. (ii) For $\mathbf{A}^{k+2} = \rho^2\mathbf{A}^k$, $\rho \neq 0$, $k = 2l$ $(l \in \mathbf{N}_0)$, we have

$$\begin{aligned} e^{\mathbf{A}} &= \sum_{i=0}^{\infty} \frac{\mathbf{A}^i}{i!} \\ &= \sum_{i=0}^{k-1} \frac{\mathbf{A}^i}{i!} + \left(\sum_{m=k/2}^{\infty} \frac{\rho^{2m-k}}{(2m)!}\right)\mathbf{A}^k + \left(\sum_{m=k/2}^{\infty} \frac{\rho^{2m-k}}{(2m+1)!}\right)\mathbf{A}^{k+1} \end{aligned}$$