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Initial value problem with infinitely many linear-like solutions for a second-order differential equation

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Abstract

A multiplicity result for an initial value problem is established via reduction to a first-order differential equation.

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1. Introduction

In a series of recent papers [1–10], the authors have established the existence of solutions to the second-order nonlinear differential equation

$$u'' = f(t, u, u'), \qquad t \geq t_0 \geq 1, \tag{1}$$

that behave asymptotically, in rather general circumstances, in the same way as straight lines: either (see [2,3,11]) the solution can be represented like

$$u(t) = at + o(t) \quad \text{as } t \to +\infty, \tag{2}$$

or (see [9,10])

$$u(t) = at + b + o(1) \qquad \text{as } t \to +\infty, \tag{3}$$

where $a, b \in \mathbb{R}$. A study of these solutions, usually called *linear-like* [7] or *asymptotically linear* [10], is of substantial importance for the oscillation theory of ordinary and functional differential equations

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(see the references in [9]) as well as for the existence theory for positive solutions of elliptic problems in exterior domains (see [11,12]). The same behavior was investigated recently in connection with Weyl's limit circle/limit point classification of differential operators in the theory of singular Sturm–Liouville problems (see [13]).

In the paper [14], investigating a particular case of Eq. (1), the author has demonstrated the existence of a solution $u(t)$ with asymptotic representation (2) by employment of the Green's function associated with the linear part of Eq. (1) (cf. also [8]) and the Schauder–Tikhonov fixed point theorem. As stressed in the introductory part of [14], there are virtually no multiplicity results either for boundary value problems on the positive half-line or for initial value problems having solutions that can be expressed for all large t like in (2) and (3).

In this note, using a technique that was applied successfully to similar problems by the author and Yu. Rogovchenko [15], we give a positive partial answer to the following (open) Lavrentiev–Tamarkin type of problem (see [16 (p. 98),17,18,19 (II.5)]): is there a continuous function $f : [t_0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ such that, for *every* choice of initial point (t_0, u_0, v_0), the initial value problem

$$\begin{cases} u'' = f(t, u, u'), & t \geq t_0 \geq 1, \\ u(t_0) = u_0 & u'(t_0) = v_0 \end{cases} \tag{4}$$

has an infinity of solutions defined in $[t_0, +\infty)$ that can be represented asymptotically by either (2) or (3)? Our result is partial since the construction of such a function $f(t, u, u')$ will validate the problem only for those points (t_0, u_0, v_0) that satisfy a supplementary condition:

$$\Phi(t_0, u_0, v_0) = t_0 v_0 - u_0 = c \in \mathbb{R}. \tag{5}$$

The proof of our result relies on a theorem by Wallach (see [20,19 (p. 33),21 (p. 55)]).

2. The result

Theorem 1. *Assume that the nonlinearity $f(t, u, u')$ in problem (4) can be written as*

$$f(t, u, u') = \frac{1}{t} g(tu' - u), \qquad t \geq t_0, \tag{6}$$

where $g \in C(\mathbb{R}, \mathbb{R})$, $g(c) = g(3c) = 0$ and $g(x) > 0$ for all $x \neq c, 3c$. Suppose further that

$$\int_{c+}^{2c} \frac{\mathrm{d}x}{g(x)} < +\infty \qquad \int_{2c}^{(3c)-} \frac{\mathrm{d}x}{g(x)} = +\infty. \tag{7}$$

Then, the initial value problem (4), where t_0, u_0, v_0 verify (5), has an infinity of solutions $u(t)$, defined in $[t_0, +\infty)$, such that (3) holds.

Proof. Let us introduce in problem (4), as a new variable [15], the wronskian $w(t)$ of the linear part of Eq. (1):

$$w(t) = \begin{vmatrix} t & u(t) \\ 1 & u'(t) \end{vmatrix} = tu'(t) - u(t), \qquad t \geq t_0. \tag{8}$$

The problem (4) can be reformulated in these circumstances as below:

$$w' = g(w), t \geq t_0, \qquad w(t_0) = c. \tag{9}$$