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Shape preserving approximation in vector ordered spaces☆

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Abstract

The aim of this note is to extend some classical results on the shape preserving approximation of real functions (of real variables) to functions with values in ordered vector spaces.

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1. Introduction

Like in the case of real valued functions, firstly we can introduce the following concepts.

Definition 1.1. (i) Let $(X, \|\cdot\|)$ be a real normed space.

A generalized algebraic polynomial of degree $\le n$, with coefficients in X is an expression of the form $P_n(x) = \sum_{k=0}^{n} c_k x^k$, where $c_k \in X$, $k = 0, \ldots, n$, and $x \in [a, b]$.

(ii) For $f : [-1, 1] \to X$, the uniform kth Ditzian–Totik modulus of smoothness of f on $[-1, 1]$ is given by

$$\omega_{\phi}^{k}(f; \delta)_{+\infty} = \sup_{0 \le h \le \delta} \|\overline{\Delta}_{h\phi(x)}^{k} f(x)\|_{+\infty},$$

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where $\phi^2(x) = 1 - x^2$ and $\overline{\Delta}_h^k f(x) = \sum_{j=0}^{k}(-1)^j \binom{k}{j} f(x + kh/2 - jh)$, if $x, x \pm kh/2 \in [-1, 1]$, and $\overline{\Delta}_h^k f(x) = 0$, otherwise. Here $\| f\|_\infty = \sup\{\| f(x)\|; x \in [-1, 1]\}$.

(iii) For $f : [-1, 1] \to X$, the uniform kth modulus of smoothness is given by

$$\omega_k(f; \delta)_\infty = \sup_{0\le h\le\delta} \{\sup\{\|\Delta_h^k f(x)\|; x, x + kh \in [-1, 1]\}\}.$$

Here $\Delta_h^k f(x) = \sum_{j=0}^{n}(-1)^{k-j} \binom{k}{j} f(x + jh)$.

The main result deals with some shape preserving approximation results for generalized polynomials for $f : [-1, 1] \to X$, when $(X, \| \cdot \|, \le)$ is a normed space endowed with a structure of a ordered linear space. We thus extend the classical result for the case of real functions of real variables in [3].

The main tool used in our proof is based on the following well-known result in Functional Analysis.

Theorem 1.2. *Let $(X, \| \cdot \|)$ be a normed space over the real or complex numbers and denote by X^* the conjugate space of X. Then, $\|x\| = \sup\{|x^*(x)| : x^* \in X^*, |||x^*||| \le 1\}$, for all $x \in X$.*

2. Shape preserving approximation

In this section, $(X, \| \cdot \|, \le_X)$ is a normed space such that $\le_X$ is an order relation on X which satisfies the conditions

$x \le_X y, 0 \le \alpha,$ imply $\alpha x \le_X \alpha y$;

$x \le_X y$ and $u \le_X v$ imply $x + u \le_X y + v$.

Definition 2.1. Let $f : [a, b] \to X$.

(i) f is called increasing on $[a, b]$ if $x \le y$ implies $f(x) \le_X f(y)$;
(ii) f is called convex on $[a, b]$ if

$$f(\lambda x + (1 - \lambda)y) \le_X \lambda f(x) + (1 - \lambda) f(y), \forall x, y \in [a, b], \lambda \in [0, 1].$$

The main result is the following:

Theorem 2.2. *There exists an absolute constant $C > 0$ such that for any convex function $f : [-1, 1] \to X$ and every $n \in \mathbb{N}$, there is a convex generalized algebraic polynomial $P_n(x)$, of degree $\le n$, such that*

$$\| f - P_n\|_\infty \le C\omega_\phi^2(f; 1/n)_\infty$$

and

$$\| f(x) - P_n(x)\| \le C\omega^2 \left(f; \sqrt{1 - x^2}/n\right)_\infty, \qquad x \in [-1, 1].$$

If, in addition, f is increasing, then so is P_n.

Proof. As suggested by the case of real valued functions (see [3]), let us define the generalized algebraic polynomial of degree $\le n$ by

$$P_n(f)(x) = f(-1) + \sum_{j=0}^{n-1} s_j[R_j(x) - R_{j+1}(x)],$$