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Boundary layers for stress diffusive perturbation in viscoelastic fluids

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Abstract

In this paper, we study a stress diffusive perturbation of the system describing a viscoelastic flow. We analyse the boundary layer which arises near the boundary and we observe in particular that there is no boundary layer on the velocity at the first order.

Keywords: Boundary layer; Viscoelastic fluid; Oldroyd model; Diffusive perturbation

1. Introduction

We study the asymptotic behavior of the solutions of the Oldroyd viscoelastic model when an additive coefficient of stress diffusion goes to zero. The problem models the flow of a viscoelastic incompressible fluid. It is considered on an open and regular domain $\Omega \subset \mathbb{R}^3$ whose boundary is noted Γ. The model we consider contains an additional stress diffusion term which derives from a microscopic dumbbell analysis, see [7]. This perturbation is often present for the determination of shear banding flow, see [12]. For the mathematics study of such a model, the presence of a diffusive term can be interesting, see [3]. More generally, if for theoretical, numerical or physical reasons we need to add such a term, we prove

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here that such an addition does not basically influence the solution. In order to highlight the dependence in this coefficient of stress diffusion ε, we write the model in the form:

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \Delta u^\varepsilon + \nabla p^\varepsilon = \operatorname{div} \sigma^\varepsilon, & \operatorname{div} u^\varepsilon = 0, \\ \partial_t \sigma^\varepsilon + u^\varepsilon \cdot \nabla \sigma^\varepsilon + g(\sigma^\varepsilon, \nabla u^\varepsilon) + \sigma^\varepsilon - \varepsilon \Delta \sigma^\varepsilon = D(u^\varepsilon), & \\ u^\varepsilon(0) = u_{\text{init}}, \qquad \sigma^\varepsilon(0) = \sigma_{\text{init}}, & \end{cases} \tag{1}$$

with Neumann boundary conditions for the stress and Dirichlet for the velocity:

$$\left.\frac{\partial \sigma^\varepsilon}{\partial n}\right|_\Gamma = 0, \qquad u^\varepsilon|_\Gamma = 0. \tag{2}$$

Moreover, the bilinear function $g(\sigma, \nabla u)$ is defined by:

$$g(\sigma, \nabla u) = -W(u).\sigma + \sigma.W(u) - a(D(u).\sigma + \sigma.D(u)), \qquad a \in [-1, 1],$$

where $D(u)$, $W(u)$ respectively represent the deformation and vorticity tensors.

It is known that such a system admits a solution (see [3,5,6]). Our goal is to describe the behavior of this solution $(u^\varepsilon, p^\varepsilon, \sigma^\varepsilon)$ when the viscosity ε goes to zero. We show that the solution converges strongly in L^2 (and in fact in any space which the boundary condition $\partial_n \sigma^\varepsilon|_\Gamma = 0$ does not appear) towards (u_0, p_0, σ_0), solution of the system without the stress diffusive term:

$$\begin{cases} \partial_t u_0 + u_0 \cdot \nabla u_0 - \Delta u_0 + \nabla p_0 = \operatorname{div} \sigma_0, & \operatorname{div} u_0 = 0, \\ \partial_t \sigma_0 + u_0 \cdot \nabla \sigma_0 + g(\sigma_0, \nabla u_0) + \sigma_0 = D(u_0), & \\ u_0(0) = u_{\text{init}}, \qquad \sigma_0(0) = \sigma_{\text{init}}, \qquad u_0|_\Gamma = 0. & \end{cases} \tag{3}$$

There again, it is already shown that such a system admits a solution (see [4,8,11]).

To recover the boundary condition Eq. (2) on σ, the solution of Eq. (1) oscillates very quickly close to the boundary converging toward Eq. (3). Here, we analyse the above-mentioned generated boundary layer. Previous studies have already been undertaken on such phenomena but in different physical cases, see [1,9,10] or [13]. It is known in particular that if the boundary is characteristic ($u^\varepsilon.n|_\Gamma = 0$) then the size of the generated boundary layer is of order $\sqrt{\varepsilon}$.

2. Statements of the results

The main result is the following

Theorem 2.1. *Assume* $u_{\text{init}} \in H^4(\Omega)$ *verifies* $\operatorname{div}(u_{\text{init}}) = 0$ *and* $u_{\text{init}}.n|_\Gamma = 0$, *and* $\sigma_{\text{init}} \in H^4(\Omega)$ *then there exists* $T > 0$ *and two functions* $P \in L^\infty(0, T; H^1(\Omega \times \mathbb{R}^+)) \cap L^2(0, T; H^2(\Omega \times \mathbb{R}^+))$ *and* $\Sigma \in L^\infty(0, T; H^2(\Omega \times \mathbb{R}^+))$ *such that, on* $[0, T] \times \Omega$, *we have*

$$\begin{cases} u^\varepsilon(t, x) = u_0(t, x) + \sqrt{\varepsilon} w(t, x), \\ p^\varepsilon(t, x) = p_0(t, x) + \sqrt{\varepsilon} P\left(t, x, \dfrac{\mathrm{d}(x)}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon} q(t, x), \\ \sigma^\varepsilon(t, x) = \sigma_0(t, x) + \sqrt{\varepsilon}\, \Sigma\left(t, x, \dfrac{\mathrm{d}(x)}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon} \tau(t, x). \end{cases}$$

where $\mathrm{d}(x)$ *represents the distance from* $x \in \Omega$ *to the boundary* Γ. *The functions* w, q *and* τ *verify:*

$$w \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),$$
$$q \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),$$