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The cracked-beam problem solved by the boundary approximation method

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Abstract

The cracked-beam problem, as a variant of Motz's problem, is discussed, and its very accurate solution in double precision is explicitly provided by the boundary approximation method (BAM) (i.e., the Trefftz method). Half of its expansion coefficients are zero, which is supported by an a posteriori analysis. Finding a good model of singularity problems is important for studying numerical methods. As a singularity model, the cracked-beam problem given in this paper seems to be superior to Motz's problem in Li et al. [Z.C. Li, R. Mathon, P. Serman, Boundary methods for solving elliptic problem with singularities and interfaces, SIAM J. Numer. Anal. 24 (1987) 487–498].

MSC: 65N10; 65N30

Keywords: Boundary approximation method; Trefftz method; Motz's problem; Cracked beam problem; Singularity problems; Posteriori analysis

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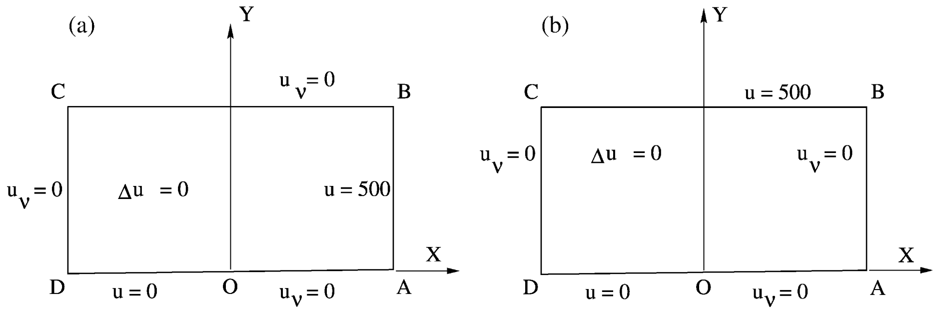

Fig. 1. Two models of Laplace's equation with singularities, (a) the Motz's problem, (b) the cracked-beam problem.

1. Motz's problem

The singularity problems of elliptic equations have drawn much attention in the last several decades. Variant numerical methods have been studied, and reported in many papers. It is important to find a typical singularity problem such that different methods may be compared with each other in numerical performance, to expose their merits and drawbacks [3,4,10]. Motz's problem [9] is a benchmark of singularity problems, which solves the Laplace equation on the rectangle $S = \{(x, y), -1 < x < 1, 0 < y < 1\}$. Let the four corners of S be $A(1, 0)$, $B(1, 1)$, $C(-1, 1)$ and $D(-1, 0)$. The mixed type of Dirichlet and Neumann conditions is enforced on its boundary (see Fig. 1(a)),

$$u\big|_{\overline{AB}} = 500, \qquad u\big|_{\overline{OD}} = 0, \qquad u_\nu\big|_{\overline{OA}} = 0, \qquad u_\nu|_{\overline{BC}\cup\overline{CD}} = 0, \tag{1}$$

where O is the origin, and u_ν is the solution derivative along the outward normal. Many methods have been developed to compute its approximate solutions. In Li et al. [6], the boundary approximation method (BAM) (i.e., the Trefftz method) is proposed to provide the very accurate solution under double precision, which is expressed as

$$v_N = \sum_{\ell=0}^{N} \tilde{D}_\ell r^{\ell+\frac{1}{2}} \cos\left(\ell + \frac{1}{2}\right)\theta, \tag{2}$$

where (r, θ) are the polar coordinates at the origin, $N = 34$, and the 35 coefficients $\tilde{D}_\ell$ are explicitly listed in [6], while an error of D_{31} was pointed out by Lucas and Oh in [8]. The approximation (2) converges to the true solution exponentially; the notorious condition number of the associated matrix also grows exponentially. To reduce the condition number, we may choose piecewise particular solutions, and apply the BAM along the interior and exterior boundary. As a consequence, the condition number decreases, but the errors increase. A strict analysis is given in [6]. Besides, the conformal transformation method (CTM) in [11] provides the very accurate 20 leading coefficients under double precision; the first 100 coefficients by CTM using Mathematica are published in [5], and the first 500 coefficients are collected in http://www.math.nsysu.edu.tw/u/scicomp/ttlu/computing.html.

2. The cracked-beam problem

When the boundary conditions on $\overline{AB}$ and on $\overline{BC}$ are exchanged as in [2,10,12], see Fig. 1(b),

$$u\big|_{\overline{BC}} = 500, \qquad u\big|_{\overline{DO}} = 0, \qquad u_\nu\big|_{\overline{OA}} = 0, \qquad u_\nu|_{\overline{AB}\cup\overline{CD}} = 0, \tag{3}$$