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Noether and master symmetries for the Toda lattice

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Abstract

In this letter we examine the interrelation between Noether symmetries, master symmetries and recursion operators for the Toda lattice. The topics include invariants, higher Poisson brackets and the various relations they satisfy. For the case of two degrees of freedom we prove that the Toda lattice is super-integrable.

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1. Introduction

In this letter we illustrate some connections between two major approaches to studying integrable Hamiltonian systems. The first approach uses Lie group analysis to obtain symmetries of differential equations. The second approach uses standard techniques in the theory of integrable systems which include Lax pair formulations, master symmetries and Poisson brackets. The main link between the two approaches is Noether's theorem. Noether proved that a physical system whose Lagrangian is invariant with respect to the symmetry transformation of a Lie group has a conservation law (constant of motion) for each generator. This is of course the main problem in Hamiltonian integrable systems: the search for invariants. If enough invariants exist, then the system is integrable and classical results allow the integration of the equations of motion. The available techniques are limited; we mention the Hamilton–Jacobi method of separation of variables, Painleve analysis, the Lax pair approach and

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Noether's theorems. In recent work, the bi-Hamiltonian approach of Magri [1] has been proven useful in conjunction with the method of separation of variables; the related concept of master symmetries is useful in the Lax approach. In this paper we show that the master symmetries may be generated from the knowledge of the hierarchy of Poisson brackets and a master integral that is obtained via Noether's theorem. We have chosen to illustrate our results on the most typical system, the Toda lattice. However, the procedure is general and may be used for other integrable systems as well. In this letter we also examine point and generalized Noether symmetries, and for the case of two degrees of freedom we produce a new integral which is not polynomial. Based on this two dimensional example we conjecture that the Toda lattice is super-integrable. A Hamiltonian system with N degrees of freedom is called super-integrable if it possesses $2N - 1$ integrals of motion.

2. Toda lattice

The Toda lattice is a Hamiltonian system with Hamiltonian function

$$H(q_1, \ldots, q_N, p_1, \ldots, p_N) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} \mathrm{e}^{q_i - q_{i+1}}. \tag{1}$$

The function $q_j(t)$ is the position of the jth particle and $p_j(t)$ is the corresponding momentum. This is the classical, finite, nonperiodic Toda lattice. This system was investigated in [2–6] and numerous other papers that are impossible to list here.

Let J_1 be the symplectic bracket with Poisson matrix

$$J_1 = 4 \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the $N \times N$ identity matrix. We define J_2 to be the tensor

$$J_2 = 2 \begin{pmatrix} A & B \\ -B & C \end{pmatrix},$$

where A is the skew-symmetric matrix defined by $a_{ij} = 1 = -a_{ji}$ for $i < j$, B is the diagonal matrix $(-p_1, -p_2, \ldots, -p_N)$ and C is the skew-symmetric matrix whose non-zero terms are $c_{i,i+1} = -c_{i+1,i} = \mathrm{e}^{q_i - q_{i+1}}$ for $i = 1, 2, \ldots, N - 1$. This bracket is due to Das and Okubo [7]; see also [8].

We define

$$h_1 = -2(p_1 + p_2 + \cdots + p_N),$$

and h_2 to be the Hamiltonian H. Then we have the following bi-Hamiltonian pair:

$$J_1 \nabla h_2 = J_2 \nabla h_1.$$

We define the recursion operator $\mathcal{R} = J_2 J_1^{-1}$ and the higher order Poisson tensors $J_i = \mathcal{R}^{i-1} J_1$. We finally define the conformal symmetry

$$Z_0 = \sum_{i=1}^{N} (N - 2i + 1) \frac{\partial}{\partial q_i} + \sum_{i=1}^{N} p_i \frac{\partial}{\partial p_i}, \tag{2}$$