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A new class of Salagean-type harmonic univalent functions

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Abstract

We define and investigate a new class of Salagean-type harmonic univalent functions. We obtain coefficient conditions, extreme points, distortion bounds, convex combination and radii of convex for the above class of harmonic univalent functions.

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1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain $\mathfrak{D}$ is said to be harmonic in $\mathfrak{D}$ if both u and v are real harmonic in $\mathfrak{D}$. In any simply connected domain we can write $f = h + \bar{g}$, where h and g are analytic in $\mathfrak{D}$. A necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathfrak{D}$ is that $|h'(z)| > |g'(z)|$, $z \in \mathfrak{D}$.

Denote by S_H the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1}$$

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In 1984 Clunie and Sheil-Small [2] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.

The differential operator D^m was introduced by Salagean [5]. For $f = h + \bar{g}$ given by (1), Jahangiri et al. [4] defined the modified Salagean operator of f as

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)} \tag{2}$$

where

$$D^m h(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k \qquad \text{and} \qquad D^m g(z) = \sum_{k=1}^{\infty} k^m b_k z^k.$$

For $0 \le \alpha < 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $z \in U$, we let $S_H(m, n; \alpha)$ denote the family of harmonic functions f of the form (1) such that

$$\operatorname{Re}\left\{\frac{D^m f(z)}{D^n f(z)}\right\} > \alpha \tag{3}$$

where $D^m f$ is defined by (2).

We let the subclass $\bar{S}_H(m, n; \alpha)$ consist of harmonic functions $f_m = h + \bar{g}_m$ in $\bar{S}_H(m, n; \alpha)$ so that h and g_m are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} b_k z^k; \quad a_k, b_k \ge 0. \tag{4}$$

The class $\bar{S}_H(m, n; \alpha)$ includes a variety of well-known subclasses of S_H. For example, $\bar{S}_H(1, 0; \alpha) \equiv \mathcal{F}(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are starlike of order α in U, $\bar{S}_H(2, 1; \alpha)$ is the class of sense-preserving, harmonic univalent functions f which are convex of order α in U, and $\bar{S}_H(n + 1, n; \alpha) \equiv \bar{H}(n, \alpha)$ is the class of Salagean-type harmonic univalent functions.

For the harmonic functions f of the form (1) with $b_1 = 0$, Avcı and Zlotkiewicz [1] showed that if $\sum_{k=2}^{\infty} k^2(|a_k| + |b_k|) \le 1$ then $f \in HK$, and Silverman [6] proved that the above coefficient condition is also necessary if $f = h + \bar{g}$ has negative coefficients. Later, Silverman and Silvia [7] improved the results of [1,6] to the case b_1 not necessarily zero.

For the harmonic functions f of the form (4) with $m = 1$, Jahangiri [3] showed that $f \in \mathcal{F}(\alpha)$ if and only if $\sum_{k=2}^{\infty}(k - \alpha)|a_k| + \sum_{k=1}^{\infty}(k + \alpha)|b_k| \le 1 - \alpha$ and $f \in \bar{S}_H(2, 1; \alpha)$ if and only if $\sum_{k=2}^{\infty} k(k - \alpha)|a_k| + \sum_{k=1}^{\infty} k(k + \alpha)|b_k| \le 1 - \alpha$. In this note, we extend the above results to the families $S_H(m, n; \alpha)$ and $\bar{S}_H(m, n; \alpha)$. We also obtain extreme points, distortion bounds, convolution conditions, and convex combinations for $\bar{S}_H(m, n; \alpha)$.

2. Main results

We begin with a sufficient coefficient condition for functions in $S_H(m, n; \alpha)$.

Theorem 1. *Let $f = h + \bar{g}$ be so that h and g are given by (1). Furthermore, let*

$$\sum_{k=1}^{\infty}\left(\frac{k^m - \alpha k^n}{1-\alpha}|a_k| + \frac{k^m - (-1)^{m-n}\alpha k^n}{1-\alpha}|b_k|\right) \le 2 \tag{5}$$