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Surface vibrational modes in disk-shaped resonators

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ABSTRACT

The natural frequencies and distributions of displacement components for the surface vibrational modes in thin isotropic elastic disks are calculated. In particular, the research is focused on even solutions for low-lying resonant vibrations with large angular wave numbers. Several families of modes are found which are interpreted as modified surface modes of an infinitely long cylinder and Lamb modes of a plate. The results of calculation are compared with the results of the experimental measurements of vibrational modes generated by means of resonant excitation in duraluminum disk with radius of ≈90 mm and thickness of 16 mm in the frequency range of 130–200 kHz. An excellent agreement between the calculated and measured frequencies is found. Measurements of the structure of the resonant peaks show splitting of some modes. About a half of the measured modes has splitting $\Delta f_{split}/f_{mode}$ at the level of the order of 10^{-5}. The *Q*-factors of all modes measured in vacuum lie in the interval $(2\ldots3) \times 10^5$. This value is typical for duraluminum mechanical resonators in the ultrasonic frequency range.

1. Introduction

High quality factor (*Q*) microsphere, microtoroid and microdisk optical resonators with whispering gallery (WG) modes have received increasing interest in recent years in a wide range of applications [1,2]. Similar to the case of the optical resonators, one may expect numerous applications of mechanical resonators with surface vibrational modes. They can be used as signal filters for telecommunication and as biological and chemical sensors [3]. The sensitivity to surface layers makes them suitable for materials science related measurements and nondestructive testing. With the development of cavity optomechanics the growing interest to the interactions between high-*Q* optical and mechanical surface modes and to the possible usage of such interactions in various applications has appeared [4].

In nonpiezoelectrical materials broadband surface acoustic waves are usually generated thermoelastically by short laser pulses [5,6]. Broadband surface acoustic waves including cylindrical Rayleigh and WG modes propagating along cylindrical surfaces were investigated theoretically and experimentally [7].

In this paper the results of the experimental study of surface vibrational modes with large angular wave numbers generated by means of resonant electrostatic excitation in a thin duraluminum disk are presented. The results of measurements are compared with the results of theoretical calculation of the natural frequencies and distributions of the displacement components for the surface vibrational modes in thin isotropic elastic disks. This allows us to identify the types of surface modes to which the measured modes belong. The *Q*-factors of the modes were measured in vacuum in order to exclude gas damping. Splitting of the resonant peaks was found for some modes.

2. Theoretical analysis

Eigenfrequencies and mode shapes of vibrating cylinders can be calculated using different methods. Among the numerical methods one of the most wide-spread is the Ritz method [8], which has recently been used to study axisymmetric vibrations of short cylinders [9], flexural vibrations of cylinders under axial loads with angular wave number of $n = 1$ [10] and vibrations of cylinders with V-notches and sharp radial cracks [11]. Three-dimensional finite difference time domain method was used to study excitation and frequency response of WG modes in disk-shaped resonators [12].

In order to calculate eigenfrequencies and mode shapes of vibrating a vibrating circular cylinder with free boundaries the analytic method developed by various authors [13,14] has been used. Hutchinson [14] obtained the frequency spectra of vibrational modes with angular wave numbers from $n = 0$ to $n = 4$ with thickness-to-diameter ratio h varying in the range of $0 < h < 2$. This method was also used in a recent work by Tamura [15], where the vibrations of a circular cylinder with high angular wave numbers n were studied. In Tamura's work the vibrations of a thick cylinder with thickness-to-diameter ratio $h \sim 1$ were mainly considered, but the case of a thin disk with $h = 0.05$ was also briefly described. In particular, the eigenfrequencies and mode shapes of vibrational modes with angular wave number $n = 20$ and the lowest resonant

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frequencies were calculated. Recently several modifications of this method were developed: to determine vibration response of a cylinder subjected to arbitrary distribution of axisymmetric excitation on its surfaces [16] and to study the vibrations of hollow cylinders with free and partially fixed boundaries [17].

In our calculations the method developed by Hutchinson is used. It involves the construction of the solutions as linear combinations of exact solutions of the differential equations of motion in three series which satisfy three of the six boundary conditions. The remaining three boundary conditions are then satisfied by orthogonalization of the corresponding stress tensor components on the boundaries. This leads to a homogeneous system of linear equations, from which the values of eigenfrequencies and the exact values of constant coefficients in solution series are calculated. The solution converges as more terms in the series are considered.

The calculation result is thereby an approximate solution which identically satisfies the differential equations and approximately satisfies the boundary conditions. Nevertheless, some authors [14] prefer to classify this method as an "exact" one, so far as it allows to construct exact infinite series solutions.

The calculation method is described in detail in Section 2.1 in order to provide the readership with a complete implementation-ready algorithm of calculation of resonant frequencies and displacement vector distributions for the vibrational modes of isotropic disks.

2.1. Method of computation

The freely vibrating disk is considered as a circular cylinder with radius a and height (thickness) $2H$ and with stress–free boundaries.

The equation of motion for the displacement vector $\mathbf{U}$ in isotropic elastic medium is

$$\rho\ddot{\mathbf{U}} = (\lambda + 2\mu)\nabla(\nabla\cdot\mathbf{U}) - \mu\nabla\times(\nabla\times\mathbf{U}), \tag{1}$$

where λ and μ are the Lame coefficients and ρ is the mass density.

We introduce the dimensionless (divided by disk radius a) components of displacement vector $\mathbf{u} = \{u_r, u_\theta, u_z\}$ and coordinates r, z; the dimensionless stress tensor components (divided by the shear modulus μ) σ_{ik}; dimensionless wave numbers (multiplied by the disk radius a) α, β and δ; dimensionless angular frequency ω that is made dimensionless by multiplying angular frequency Ω by the disk radius a and dividing by the transverse wave velocity $v_t = \sqrt{\mu/\rho}$. Dimensionless thickness parameter is thickness-to-diameter ratio $h = H/a$.

The six boundary conditions are

$$\sigma_{zr}(r,\theta,\pm h) = 0, \tag{2a}$$
$$\sigma_{z\theta}(r,\theta,\pm h) = 0, \tag{2b}$$
$$\sigma_{zz}(r,\theta,\pm h) = 0, \tag{2c}$$
$$\sigma_{rr}(1,\theta,z) = 0, \tag{2d}$$
$$\sigma_{r\theta}(1,\theta,z) = 0, \tag{2e}$$
$$\sigma_{rz}(1,\theta,z) = 0. \tag{2f}$$

We choose the three boundary conditions (2a), (2b) and (2f) to be satisfied identically. This is an arbitrary choice, the only restriction of the following method is that we cannot satisfy three boundary conditions on one boundary identically.

The solution is constructed as a linear combination of basic solutions [14]:

$$v(r,\theta,z) = \sum_{i=1}^{N_z} A_i v_{Ai}(r,\theta,z) + \sum_{j=1}^{N_r} B_j v_{Bj}(r,\theta,z) + \sum_{i=1}^{N_z} C_i v_{Ci}(r,\theta,z), \tag{3}$$

where v stands for displacement vector components u_r, u_θ, u_z and stress tensor components $\sigma_{\mu\nu}$ with the same three series of constants A_i, B_j and C_i. We exclude the time dependence by assumption that all components of the displacement vector and all components of the stress tensor vary in time sinusoidally with the same phase and frequency.

Explicitly, the radial displacement is taken in the following form:

$$u_{r_{Ai}} = [2(\beta_{Ai}^2 - \alpha_{Ai}^2)\chi_n(\beta_{Ai})\chi_n(\delta_{Ai}r) + 4\alpha_{Ai}^2\chi_n(\delta_{Ai})\chi_n(\beta_{Ai}r)] \times r^{n-1}\left\{\begin{matrix}\cos(\alpha_{Ai}z)/\cos(\alpha_{Ai}h)\\ \sin(\alpha_{Ai}z)/\sin(\alpha_{Ai}h)\end{matrix}\right\}\cos n\theta, \tag{4}$$

$$u_{r_{Bj}} = \left[2(\alpha_{Bj}^2 - \beta_{Bj}^2)\left\{\begin{matrix}\cos(\delta_{Bj}z)\mathrm{sinc}(\beta_{Bj}h)\\ \mathrm{sinc}(\delta_{Bj}z)\cos(\beta_{Bj}h)\end{matrix}\right\} + 4\left\{\begin{matrix}\delta_{Bj}^2\cos(\beta_{Bj}z)\mathrm{sinc}(\delta_{Bj}h)\\ \beta_{Bj}^2\mathrm{sinc}(\beta_{Bj}z)\cos(\delta_{Bj}h)\end{matrix}\right\}\right] \times \left\{\begin{matrix}h\\ z\end{matrix}\right\} r^{n-1}\frac{\chi_n(\alpha_{Bj}r)}{\phi_n(\alpha_{Bj})}\cos n\theta, \tag{5}$$

$$u_{r_{Ci}} = [-4\chi_n(\delta_{Ci})\phi_n(\beta_{Ci}r) + 2\phi_n(\beta_{Ci})\chi_n(\delta_{Ci}r)] \times nr^{n-1}\left\{\begin{matrix}\cos(\alpha_{Ci}z)/\cos(\alpha_{Ci}h)\\ \sin(\alpha_{Ci}z)/\sin(\alpha_{Ci}h)\end{matrix}\right\}\cos n\theta; \tag{6}$$

for the axial displacement we take (we omit the indices i and j for simplicity; they relate to the terms u, α, β and δ in the same way as in the previous three equations)

$$u_{z_A} = [2(\beta_A^2 - \alpha_A^2)\chi_n(\beta_A)\phi_n(\delta_A r) - 4\beta_A^2\chi_n(\delta_A)\phi_n(\beta_A r)] \times \alpha_A r^n\left\{\begin{matrix}-\sin(\alpha_A z)/\cos(\alpha_A h)\\ \cos(\alpha_A z)/\sin(\alpha_A h)\end{matrix}\right\}\cos n\theta, \tag{7}$$

$$u_{z_B} = \left[2(\alpha_B^2 - \beta_B^2)\left\{\begin{matrix}\mathrm{sinc}(\delta_B z)\mathrm{sinc}(\beta_B h)\\ \cos(\delta_B z)\cos(\beta_B h)\end{matrix}\right\} - 4\alpha_B^2\left\{\begin{matrix}\mathrm{sinc}(\delta_B h)\mathrm{sinc}(\beta_B z)\\ \cos(\delta_B h)\cos(\beta_B z)\end{matrix}\right\}\right] \times \left\{\begin{matrix}-\delta_B^2 hz\\ 1\end{matrix}\right\} r^n\frac{\phi_n(\alpha_B r)}{\phi_n(\alpha_B)}\cos n\theta, \tag{8}$$

$$u_{z_C} = 2n\phi_n(\beta_C)\phi_n(\delta_C r)\alpha_C r^n \times \left\{\begin{matrix}-\sin(\alpha_C z)/\cos(\alpha_C h)\\ \cos(\alpha_C z)/\sin(\alpha_C h)\end{matrix}\right\}\cos n\theta; \tag{9}$$

and for the angular displacement we take

$$u_{\theta_A} = -[2(\beta_A^2 - \alpha_A^2)\chi_n(\beta_A)\phi_n(\delta_A r) + 4\alpha_A^2\chi_n(\delta_A)\phi_n(\beta_A r)] \times nr^{n-1}\left\{\begin{matrix}\cos(\alpha_A z)/\cos(\alpha_A h)\\ \sin(\alpha_A z)/\sin(\alpha_A h)\end{matrix}\right\}\sin n\theta, \tag{10}$$

$$u_{\theta_B} = -\left[2(\alpha_B^2 - \beta_B^2)\left\{\begin{matrix}\cos(\delta_B z)\mathrm{sinc}(\beta_B h)\\ \mathrm{sinc}(\delta_B z)\cos(\beta_B h)\end{matrix}\right\} + 4\left\{\begin{matrix}\delta_B^2\cos(\beta_B z)\mathrm{sinc}(\delta_B h)\\ \beta_B^2\mathrm{sinc}(\beta_B z)\cos(\delta_B h)\end{matrix}\right\}\right] \times \left\{\begin{matrix}h\\ z\end{matrix}\right\} nr^{n-1}\frac{\phi_n(\alpha_B r)}{\phi_n(\alpha_B)}\sin n\theta, \tag{11}$$

$$u_{\theta_C} = [4\chi_n(\delta_C)\chi_n(\beta_C r) - 2n^2\phi_n(\beta_C)\phi_n(\delta_C r)]r^{n-1} \times \left\{\begin{matrix}\cos(\alpha_C z)/\cos(\alpha_C h)\\ \sin(\alpha_C z)/\sin(\alpha_C h)\end{matrix}\right\}\sin n\theta. \tag{12}$$

The relations between dimensionless wave numbers α, β and δ and frequency ω are

$$\alpha^2 + \beta^2 = \omega^2, \tag{13}$$

$$\alpha^2 + \delta^2 = \omega^2\frac{1-2\sigma}{2(1-\sigma)} = \left(\frac{\omega}{\eta}\right)^2, \tag{14}$$

where σ is the Poisson's ratio and $\eta = v_l/v_t$.

The functions ϕ_n and χ_n introduced in Eqs. (4)–(12) are defined as