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Short Communication

Verification of surface wave solutions obtained by the reciprocity theorem

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ABSTRACT

Surface wave motions generated by a time-harmonic point load applied at the surface of an isotropic linearly elastic half-space are conventionally solved by the use of integral transform techniques. The inverse transforms, are often complicated and will not always yield closed-form solutions. In this paper expressions for the displacements for surface wave motions radiated from point-load excitation are determined in a simple manner by the use of the elastodynamic reciprocity theorem. It is shown that the radiated amplitudes of the surface displacements obtained by the reciprocity approach are identical to the corresponding results obtained by the use of Hankel transform and by Lamb in his classical paper.

1. Introduction

The wave motion generated by application of a time-harmonic point load at the surface of an elastic half-space is a fundamental problem of elastodynamics. The solution was obtained by Lamb [1] in 1904. The problem is now known as Lamb's problem. Lamb fully discussed the wave motions generated by a line load and a point load applied normal to the surface. Explicit expressions were given for the generated surface waves, for both loads of harmonic-time dependence and impulsive loads. Lamb's method is, however, very intricate. Using Hankel and Laplace transforms, Pekeris [2] gave exact and closed-form expressions for the displacements for the case when the load varies like the Heaviside step function. Nearly simultaneously with Pekeris, the normal load problem was also treated by Pinney [3]. Subsequent discussions can be found in the books by Ewing et al. [4], Achenbach [5], and Graff [6]. In these books, Lamb's methods and solutions have been cast in a somewhat more elegant form and more detailed computations have been carried out, particularly for loads of arbitrary time dependence, based on the use of integral transform techniques. The response of an elastic half-space to a tangential surface load varying with time as the Heaviside step function was investigated by Chao [7]. Several numerical calculations for Lamb's problem have also been carried out, for example, by Mooney [8] and Chuhan et al. [9].

Solutions for Lamb's problem by integral transform techniques require inverse integral transforms which are often complicated and will not always yield closed-form solutions. Moreover, the integral transform approach becomes more difficult for anisotropic solids, and impossible for inhomogeneous solids, for example, solids whose elastic moduli depend on the depth coordinate, as in geophysical applications and functionally graded materials. To avoid these difficulties, another approach has been proposed in recent years, based on the elastodynamic reciprocity theorem, strictly to determine the surface waves, see Achenbach and Xu [10], Achenbach [11], Achenbach [12,13], and Phan et al. [14].

It is, of course, of interest to compare the results obtained by the reciprocity approach with the corresponding results obtained by the use of integral transform techniques. The classical approach shows that a normal point load on a homogeneous half-space generates body waves, their interactions which are called head waves, and classical surface waves. Sufficiently far away from the point of load application the surface waves dominate. Still it cannot be assumed that all the energy that reaches large distances ends up in surface waves only. Therefore if the far-field is a priori assumed to be only a surface wave, as is done in the application of the reciprocity theorem, it cannot be assumed without verification that the reciprocity theorem produces the same result as the classical approach which is based on a computation that includes all waves. It is indeed necessary to verify that the simple reciprocity

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approach gives the same result as Lamb's Fourier Integral approach.

The main purpose of the current paper is, therefore, to show that the method based on the elastodynamic reciprocity theorem is exceedingly simple, and most importantly gives results identical to the ones obtained by the aid of the Hankel transform and by Lamb [1]. The paper proceeds through 5 sections. In Section 2, the method to determine the displacement fields of surface waves generated by a time-harmonic point load using the reciprocity theorem is discussed. In Section 3 solutions of surface waves generated by a point load are obtained by the use of Hankel transform and shown to be mathematically the same as the results obtained by Lamb. Section 4 shows the verification of the reciprocity approach in comparison with the Hankel transform approach and the methods of Lamb's paper. Section 5 states conclusions.

2. Surface waves generated by a time-harmonic point load

It has been shown in [12] that the reciprocity theorem can be used to determine the surface wave motion generated by a time-harmonic point load applied at the surface of a half-space. The surface wave motion is calculated in a simple manner by the reciprocity theorem, with input the actual surface wave with unknown amplitude and a virtual free surface wave. The method requires expressions for the displacements and the stresses of free surface waves, preferably in analytical form, but numerically obtained forms can also be used. Fig. 1 shows a half-space of a homogeneous, isotropic, linearly elastic solid referred to Cartesian and cylindrical coordinates, such that the x_1x_2-plane coincides with the surface of the half-space. The half-space is subjected to a time-harmonic point load at the surface pointing in an arbitrary direction. Without loss of generality, the coordinate system can be chosen such that the load acts in the x_1z-plane, at the origin of the system. The surface wave response is then sought as the superposition of the responses due to the normal component P and the component Q in the x_1-direction. It is convenient to also use cylindrical coordinates (r, θ, z) defined by $x_1 = r\cos\theta$, $x_2 = r\sin\theta, z$.

In addition that its amplitude decreases with depth, a surface wave is defined by an angular frequency ω and a wavenumber k, where $k = \omega/c$, c being the surface wave velocity, as well as material properties, the Lamé constants λ and μ, and the mass density ρ.

It has been shown in [12] that in cylindrical coordinates a guided wave, such as a Rayleigh surface wave can be represented by

$$u_r = U(z)k^{-1}\frac{\partial\varphi}{\partial r}e^{-i\omega t} \tag{1}$$

$$u_\theta = U(z)(kr)^{-1}\frac{\partial\varphi}{\partial\theta}e^{-i\omega t} \tag{2}$$

$$u_z = W(z)\varphi e^{-i\omega t}, \tag{3}$$

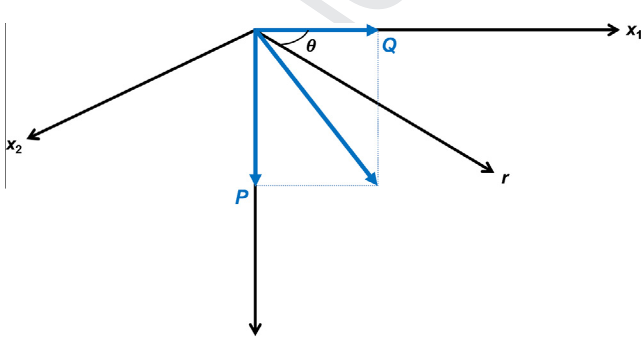

Fig. 1. Half-space subjected to a time-harmonic point load.

where for axially-symmetric Rayleigh waves $\varphi(r)$ is the solution of

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + k^2\varphi = 0 \tag{4}$$

When the Rayleigh waves are not axially symmetric, the equation for $\varphi(r,\theta)$ is

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + k^2\varphi = 0 \tag{5}$$

For both cases the functions $U(z)$ and $W(z)$ are the solutions of the same set of coupled equations whose solutions are

$$U(z) = d_1e^{-kpz} + d_2e^{-kqz} \tag{6}$$

$$W(z) = d_3e^{-kpz} - e^{-kqz}, \tag{7}$$

where

$$d_1 = \frac{-(1+q^2)}{2p},\ d_2 = q,\ d_3 = \frac{1+q^2}{2} \tag{8}$$

Here

$$p = \sqrt{1 - c^2/c_L^2},\ q = \sqrt{1 - c^2/c_T^2} \tag{9}$$

$$\text{where } c_L = \sqrt{(\lambda+2\mu)/\rho},\ c_T = \sqrt{\mu/\rho} \tag{10}$$

are the longitudinal and transverse wave velocities, respectively.

Let us first consider the case of a horizontal time-harmonic load applied at the origin on the surface of the half-space, pointing in the x_1-direction

$$f_1 = Q\delta(x_1)\delta(x_2)\delta(z)e^{-i\omega t} \tag{11}$$

It has been argued by Achenbach [12, p.135] that for this case Eq. (5) applies, and we should consider the solution

$$\varphi(r,\theta) = A_Q\Phi(kr)\cos\theta \tag{12}$$

where for an outgoing wave compatible with $\exp(-i\omega t)$ we have

$$\Phi(kr) = H_1^{(1)}(kr) \tag{13}$$

We can equally well consider an incoming wave, i.e., a wave that converges on the origin,

$$\bar{\Phi}(kr) = H_1^{(2)}(kr) \tag{14}$$

In Eqs. (13) and (14), $H_j^{(i)}(\xi)$ is the jth order Hankel function of the ith kind.

Expressions for the displacements corresponding to Eqs. (12)–(14) have been given by in [12 , pp.137-138]. For the reciprocity theorem, we consider the region V defined by $0 \leqslant r \leqslant b, 0 \leqslant z \leqslant \infty, 0 \leqslant \theta \leqslant 2\pi$. As part of the reciprocity theorem we have to determine the interaction of the force defined by Eq. (11) with the compatible displacement in the x_1-direction of the virtual wave. The virtual wave is chosen as the sum of an outgoing and a converging wave, as defined by Eqs. (8.4.5)–(8.4.7) of Achenbach [12], i.e., for u_r^B we have

$$u_r^B = \frac{1}{2}BU(z)[\Phi'(kr) + \bar{\Phi}'(kr)]\cos\theta, \tag{15}$$

where $\Phi(kr)$ and $\overline{\Phi}(kr)$ are defined by Eqs. (13) and (14), and the prime defines the derivative with respect to the argument.

The derivatives appearing in Eq. (15) can be calculated from a standard formula McLachlan [15 , p. 198, no. 128]. The details, which have been given by Hao et al. [16 Eqs. (A-32)–(A-38)], result in

$$\Phi'(\xi) + \overline{\Phi}'(\xi) = \frac{2}{\xi}J_1(\xi) - 2J_2(\xi) \tag{16}$$

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