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Short Communication

Propagation of thickness-twist waves in elastic plates with periodically varying thickness and phononic crystals

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A B S T R A C T

We study the propagation of thickness-twist (TT) waves in a crystal plate of AT-cut quartz with periodically varying, piecewise constant thickness. The scalar differential equation by Tiersten and Smythe is employed. The problem is found to be mathematically equivalent to the motion of an electron in a periodic potential field governed by Schrodinger's equation. An analytical solution is obtained. Numerical results show that the eigenvalue (frequency) spectrum of the waves has a band structure with allowed and forbidden bands. Therefore, for TT waves, plates with periodically varying thickness can be considered as phononic crystals. The effects of various parameters on the frequency spectrum are examined.

1. Introduction

It is well known that the motion of a mobile charge in a crystal is governed by Schrodinger's equation with a periodic potential, and the resulting eigenvalue spectrum has a band structure with allowed and forbidden electronic energy bands that can describe semiconduction [1]. Mathematically, this is a consequence of the periodically varying coefficients in the governing differential equation and the periodic boundary conditions. Therefore the band structure of the eigenvlaue spectrum also appears in other periodic physical systems, e.g., the propagation of electromagnetic or acoustic waves in periodically varying materials or structures called photonic [2,3] or phononic [4,5] crystals. The study of photonic and phononic crystals has led to new materials and structures with potentials for new and useful optic or acoustic wave devices. This paper is concerned with phononic crystals which have been developed for both surface [6,7] and bulk [8–12] acoustic waves. The literature on phononic crystals are abundant are growing rapidly [13–23]. In this paper we study a special class of acoustic waves in elastic plates called thickness-twist (TT) waves [24] which are widely used as the operating modes of acoustic wave resonators and sensors. Differing from most phononic crystal plates in the literature which are made from composite materials, the plate in the present analysis is made from a homogeneous material but it has a periodically changing thickness. TT waves are classified as high-frequency waves in plates. Their dispersion relations have finite cutoff frequencies below which the waves cannot propagate. This is fundamentally different from the low-frequency waves of extension and flexure whose dispersion relations do not have cutoff frequencies. We show theoretically that when an elastic plate has a periodically varying thickness, the governing equation for the high-frequency TT waves is mathematically equivalent to Schrodinger's equation with a periodic potential. The solution of the equation shows that the frequency spectrum of these waves exhibits band gaps that prevent TT phonons with selected ranges of frequencies from being transmitted through the plate.

2. Governing equations

Consider the elastic plate in Fig. 1. It is the cross section of a plate that is unbounded in the x_1 direction which is determined from x_3 and x_2 by the right-hand rule. We consider motions independent of x_1.

Specifically, for AT-cut quartz plates, motions called TT waves with only one displacement component $u_1(x_2, x_3, t)$ are allowed by the equations of anisotropic elasticity. The nth-order TT displacement $u_1^n(x_3, t)$ in an AT-cut quartz plate is defined by [25,26]

$$u_1(x_2, x_3, t) = \sum_{n=1,3,5,\ldots}^{\infty} u_1^n(x_3, t) \sin\frac{n\pi x_2}{2h}, \tag{1}$$

where $x_2 = y$ and $x_3 = z$. n is an odd integer. $2h$ is the plate thickness which may be a constant or a function of z. By studying the

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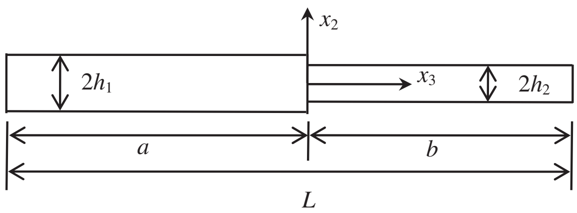

Fig. 1. Cross section of an AT-cut quartz plate with a piecewise constant thickness (unit cell of a periodic plate obtained by repeated extensions in the x_3 direction).

propagation of long u_1 waves in the z direction, it was shown [25] that u_1^n is governed by the following equation:

$$c_{55}\frac{\partial^2 u_1^n}{\partial z^2} - \frac{n^2\pi^2}{4h^2}\bar{c}_{66}u_1^n - \rho\ddot{u}_1^n = 0, \quad (2)$$

where $\bar{c}_{66} = c_{66} + e_{26}^2/\varepsilon_{66}$. c_{55} and c_{66} are two relevant elastic constants. e_{26} is a relevant piezoelectric constant.

For time-harmonic motions, we let

$$u_1^n(z,t) = \psi(z)e^{i\omega t}. \quad (3)$$

The substitution of (3) into (2) gives

$$\frac{\partial^2\psi}{\partial z^2} + \frac{M}{c_{55}}\left(H(z) + \frac{\rho\omega^2}{M}\right)\psi = 0, \quad (4)$$

where

$$M = -\frac{n^2\pi^2\bar{c}_{66}}{4}, \quad H(z) = \frac{1}{h(z)^2}. \quad (5)$$

Mathematically, (4) is equivalent to the motion of a mobile charge in a potential field governed by Schrodinger's equation [1]. Therefore the eigenvalue spectrum of (4) under periodic boundary conditions may be expected to have a band structure.

3. Solution

The solution to (4) is parallel to that in [1] for an electron in a crystal. We look for a solution in the following form [1]:

$$\psi(z) = Ae^{ikz}. \quad (6)$$

Substituting (6) into (4), we can determine k through

$$-k^2 + \frac{M}{c_{55}}\left(H(z) + \frac{\rho\omega^2}{M}\right) = 0, \quad k = \pm\sqrt{\frac{M}{c_{55}}\left(H(z) + \frac{\rho\omega^2}{M}\right)}, \quad (7)$$

where for each of the two parts of the plate in Fig. 1, the plate thickness h is a constant and hence H and k are constants. Then the solution to (4) may be written in the following form:

$$\psi(z) = Ae^{ikz} + Be^{-ikz}. \quad (8)$$

According to the well-known Bloch theorem [1], when $H(z)$ is a periodic function with a period L (see Fig. 1), (8) can be written as the product of a plane wave characterized by a propagation constant μ and a periodic function $U(z)$ with the same period L:

$$\psi(z) = e^{i\mu z}U(z), \quad (9)$$

where $U(z)$ satisfies

$$U(z) = U(z+L). \quad (10)$$

According to (9) and (10), ψ at $z+L$ can be expressed as

$$\psi(z+L) = e^{i\mu(z+L)}U(z+L) = e^{i\mu L}e^{i\mu z}U(z) = e^{i\mu L}\psi(z). \quad (11)$$

Specifically, for each of the two parts of the plate in Fig. 1, the nth-order TT displacements u_1^n in each part can be written as

$$u_1^{n1} = (A_1e^{ik_1z} + B_1e^{-ik_1z})e^{i\omega t}, \quad -a < z < 0 \quad (12)$$

and

$$u_1^{n2} = (A_2e^{ik_2z} + B_2e^{-ik_2z})e^{i\omega t}, \quad 0 < z < b, \quad (13)$$

respectively, where

$$k_j = \pm\sqrt{\frac{M_j}{c_{55}^j}\left(H_j(z) + \frac{\rho_j\omega^2}{M_j}\right)} \quad (j = 1, 2). \quad (14)$$

In (14), to distinguish the parameters of the two parts of the plate in Fig. 1, we have introduced a subscript (or superscript) $j = 1$ or 2.

At the junction between the two parts of the plate in Fig. 1 where $z = 0$, we impose the continuity of the TT displacement and the corresponding twisting moment:

$$u_1^{n1}(0,t) = u_1^{n2}(0,t),$$
$$\int_{-h_1}^{h_1} c_{55}^1 u_{1,3}^{n1}(0,t)\sin\tfrac{\pi n x_2}{2h_1}x_2 dx_2 = \int_{-h_2}^{h_2} c_{55}^2 u_{1,3}^{n2}(0,t)\sin\tfrac{\pi n x_2}{2h_2}x_2 dx_2. \quad (15)$$

In (15), we have denoted $u_{1,3} = \partial u_1/\partial z$. At the two ends of the plate in Fig. 1, we have the following periodic conditions for the TT displacement and the twisting moment:

$$u_1^{n1}(-a,t) = u_1^{n2}(b,t)e^{-i\mu L},$$
$$\int_{-h_1}^{h_1} c_{55}^1 u_{1,3}^{n1}(-a,t)\sin\tfrac{\pi n x_2}{2h_1}x_2 dx_2 = e^{-i\mu L}\int_{-h_2}^{h_2} c_{55}^2 u_{1,3}^{n2}(b,t)\sin\tfrac{\pi n x_2}{2h_2}x_2 dx_2. \quad (16)$$

The substitution of (12) and (13) into (15) and (16) yields

$$A_1 + B_1 = A_2 + B_2,$$
$$h_1^2 c_{55}^1(ik_1A_1 - ik_1B_1) = h_2^2 c_{55}^2(ik_2A_2 - ik_2B_2),$$
$$A_1e^{-ik_1a} + B_1e^{ik_1a} = (A_2e^{ik_2b} + B_2e^{-ik_2b})e^{-i\mu L},$$
$$h_1^2c_{55}^1(ik_1A_1e^{-ik_1a} - ik_1B_1e^{ik_1a}) = e^{-i\mu L}h_2^2c_{55}^2(ik_2A_2e^{ik_2b} - ik_2B_2e^{-ik_2b}). \quad (17)$$

(17) represents a system of linear homogeneous equations for A_1, B_1, A_2 and B_2:

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ k_1 & -k_1 & -\frac{h_2^2c_{55}^2}{h_1^2c_{55}^1}k_2 & \frac{h_2^2c_{55}^2}{h_1^2c_{55}^1}k_2 \\ e^{-ik_1a} & e^{ik_1a} & -e^{i(k_2b-\mu L)} & -e^{-i(k_2b+\mu L)} \\ ik_1e^{-ik_1a} & -ik_1e^{ik_1a} & -\frac{h_2^2c_{55}^2}{h_1^2c_{55}^1}ik_2e^{i(k_2b-\mu L)} & \frac{h_2^2c_{55}^2}{h_1^2c_{55}^1}ik_2e^{-i(k_2b+\mu L)} \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix} = \mathbf{0}. \quad (18)$$

For nontrivial solutions the determinant of the coefficient matrix in (18) has to vanish, which leads to the characteristic equation that determines the wave frequency ω for a given propagation constant μ.

4. Numerical results and discussion

As a numerical example, consider the case when the two parts of the plate in Fig. 1 are both of AT-cut quartz whose material constants can be found in [27]. For Figs. 2–4 the geometric parameters are fixed to be $h_1 = 2h_2 = 0.01$ m and $a = 5b = 0.1$ m. Some of them will be varied later in Figs. 5–7.

Fig. 2 shows the dimensionless frequency $\left(\Omega = \sqrt{4h_1^2\rho_1\omega^2/\pi^2\bar{c}_{66}}\right)$ versus the dimensionless propagation constant μL for the fundamental TT wave with $n = 1$ in (2). The curves vary periodically with respect to μL. Only the first Brillouin zone with $-\pi < \mu L < \pi$ is shown in the figure. The dimensionless cutoff frequency for the fundamental TT waves in the thicker part of the plate with thickness $2h_1$ is $\Omega = 1$. Since $h_1 = 2h_2$, the cutoff frequency of the fundamental TT

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