



# Elastic constants of solids and fluids with initial pressure via a unified approach based on equations-of-state



John H. Cantrell\*

Research Directorate, NASA Langley Research Center, Hampton, VA 23681, USA

## ARTICLE INFO

### Article history:

Received 2 December 2013

Accepted 14 January 2014

Available online 25 January 2014

### Keywords:

Elastic constants

Fluids

Solids

Nonlinearity parameters

Initial pressure

## ABSTRACT

The second and third-order Brugger elastic constants are obtained for liquids and ideal gases having an initial hydrostatic pressure  $p_1$ . For liquids the second-order elastic constants are  $C_{11} = A + p_1$ ,  $C_{12} = A - p_1$ , and the third-order constants are  $C_{111} = -(B + 5A + 3p_1)$ ,  $C_{112} = -(B + A - p_1)$ , and  $C_{123} = A - B - p_1$ , where  $A$  and  $B$  are the Beyer expansion coefficients in the liquid equation of state. For ideal gases the second-order constants are  $C_{11} = p_1\gamma + p_1$ ,  $C_{12} = p_1\gamma - p_1$ , and the third-order constants are  $C_{111} = -p_1(\gamma^2 + 4\gamma + 3)$ ,  $C_{112} = -p_1(\gamma^2 - 1)$ , and  $C_{123} = -p_1(\gamma^2 - 2\gamma + 1)$ , where  $\gamma$  is the ratio of specific heats. The inequality of  $C_{11}$  and  $C_{12}$  results in a nonzero shear constant  $C_{44} = (1/2)(C_{11} - C_{12}) = p_1$  for both liquids and gases. For water at standard temperature and pressure the ratio of terms  $p_1/A$  contributing to the second-order constants is approximately  $4.3 \times 10^{-5}$ . For atmospheric gases the ratio of corresponding terms is approximately 0.7. Analytical expressions that include initial stresses are derived for the material 'nonlinearity parameters' associated with harmonic generation and acoustoelasticity for fluids and solids of arbitrary crystal symmetry. The expressions are used to validate the relationships for the elastic constants of fluids.

Published by Elsevier B.V.

## 1. Introduction

Nonlinear acoustical methods are becoming increasingly popular for the nondestructive evaluation and characterization of materials [1–9]. Often the research centers on assessing the mechanical properties of multiphase materials, including composites, porous and granular materials, and solids with embedded liquid and gas phases resulting from materials processing. Evaluations are often conducted for solids immersed in a liquid bath or coupled to a transducer by a liquid jet for which the effects of the liquid must be considered. In other cases air-coupled transducers are used for which the effects of the air are important. It would thus be useful in some analyses of acoustic wave propagation in fluid–solid media to have available the linear (second-order) and nonlinear (third-order) elastic coefficients of the fluids, since such constants provide a common means of quantifying elastic behavior for both solids and fluids. Although a variety of elastic constants are defined in the literature, we consider the Brugger elastic constants for present purposes because of their symmetry properties and because they provide a direct connection with thermodynamics. The Brugger constants are defined for isentropic conditions from the internal energy and for isothermal conditions from the Helmholtz free energy [10]. The number of independent Brugger constants also reflects the symmetry group of the material, ranging from 21

second-order constants and 56 third-order constants for triclinic symmetry to two second-order constants and three third-order constants for isotropic symmetry. Fluids are treated as a special case of a material with isotropic symmetry.

Quite relevant to nondestructive materials characterization, a number of researchers have used variously defined 'nonlinearity parameters' to quantify the deviation of a mechanical system from linear stress–strain behavior. The various nonlinearity parameters are defined by the experimental techniques used to measure them and are functionally dependent on the elastic constants, even though the relationship to the elastic constants in some cases is quite complicated. More importantly for present purposes, the nonlinear elastic constants are generally obtained from measurements of the nonlinearity parameters. Fox and Wallace [11], Rudnick [12], Beyer [13], and Zarembo and Krasil'nikov [14] defined parameters related to nonlinearity in fluids. Truesdell [15], Toupin and Berstein [16], Landau and Lifshitz [17], Murnaghan [18], Brugger [10], and Mason [19] defined various nonlinear constants to describe elastic nonlinearity in bulk solids. Breazeale and Ford [20] pointed out that certain ratios of second and third-order Brugger elastic constants can be used to quantify acoustic harmonic generation along the pure mode propagation directions in initially stress-free cubic solids. Yost and Breazeale [21] defined these ratios of elastic constants as the acoustic nonlinearity parameters. Cantrell [22] subsequently proposed a slightly different but more general definition of the acoustic nonlinearity parameters for plane wave propagation of arbitrary polarization and direction

\* Tel.: +1 757 864 4989.

E-mail address: [john.h.cantrell@nasa.gov](mailto:john.h.cantrell@nasa.gov)

(propagation modes) in initially stress-free crystals of arbitrary symmetry. He found that the nonlinearity parameters are independently ordered for each mode according to the crystalline symmetry and structure of the solid and explained such dependence from a consideration of the Born–Mayer interatomic pair potentials [23].

The mechanical behavior of fluids is generally expressed by an equation of state that links the pressure with the mass density of the fluid. We seek to relate the second and third-order Brugger elastic constants to the ‘elastic parameters’ in the relevant fluid equation of state (e.g., the Beyer coefficients in the liquid equation of state). For completeness it is necessary to include the initial pressure in the derivation, since in some cases (e.g., ideal gases) the initial pressure makes a substantial contribution to the equation of state. We begin in Section 2 by including initial stresses in the Cauchy stress equation via the formalism of finite deformation theory. Since the third-order elastic constants are most often determined from the nonlinearity parameters measured via acoustic harmonic generation or from the variation of the sound velocity with applied stress (acoustoelasticity), it is instructive to obtain expressions for the nonlinearity parameters corresponding to these methods by generalizing the analytical expressions previously obtained for the nonlinearity parameters [22–25] to include the effects of initial stresses. More importantly, the generalized equations are necessary to provide a quantitative validation of the relationships to be obtained in Section 3 for the elastic constants of fluids. The distinction and relevance of the ‘natural’ and ‘true’ velocities to solids and fluids in assessing the nonlinearity parameters associated with acoustoelasticity and acoustic harmonic generation is critically discussed.

The formalism developed in Section 2 is applied in Section 3 to the treatment of fluid media by assuming that an ideal fluid can be represented as an isotropic material that will yield under the action of a shear stress, no matter how small. Relationships between the elastic constants and the ‘elastic parameters’ in the relevant equation of state are obtained for both liquids and ideal gases by comparing corresponding coefficients in the strain expansions of the Cauchy relation and the equation of state. The relationships are validated using expressions derived for the nonlinearity parameters corresponding to acoustic harmonic generation and to acoustoelasticity.

## 2. Finite deformation theory in solids with initial stress

We apply finite deformation theory to extend the definitions of the nonlinearity parameters previously defined to quantify acoustic harmonic generation [22,23] and acoustoelasticity for solids [24,25] to include the effects of initial stresses.

### 2.1. Finite deformation theory

The theory of finite deformations was originally developed by Murnaghan [18] and expanded to include acoustic wave propagation by Thurston [24], Thurston and Brugger [25], Thurston and Shapiro [26], and Wallace [27]. The initial configuration of particles comprising a material body is denoted by the set of position vectors  $\mathbf{X}$ , often called the material or Lagrangian coordinates of the material. The initial configuration is generalized here to include the effects of initial stresses in the material, if any. The present configuration of the particles is denoted by the set of position vectors  $\mathbf{x}$ , often called the spatial or Eulerian coordinates of the material. The present configuration is obtained by means of an elastic deformation defined by the set of transformation coefficients  $\alpha_{ij} = \partial x_i / \partial X_j$  for small finite deformations, where  $x_i$  and  $X_j$ , respectively, are the Cartesian components of the vectors  $\mathbf{x}$  and  $\mathbf{X}$ . The indices  $i$  and  $j$  take the values 1, 2, 3 representing three mutually orthogonal

spatial reference axes. If the deformation is nonuniform, as is the case of acoustic wave propagation, the transformation is considered to be local in  $\mathbf{X}$  and time  $t$ , i.e.  $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ .

The particle displacement  $\mathbf{u}$  is defined by  $\mathbf{u} = \mathbf{x} - \mathbf{X}$ . The transformation coefficients  $\alpha_{ij}$  are related to the displacement gradients  $u_{ij} = \partial u_i / \partial X_j$  as

$$\alpha_{ij} = \delta_{ij} + u_{ij}, \quad (1)$$

where  $\delta_{ij}$  is the Kronecker delta. The deformation results in a change in the mass density of the material given by [18]

$$\frac{\rho_1}{\rho} = J = \det[\alpha_{ij}] = 1 + u_{kk} + O(u_{ij}), \quad (2)$$

where  $\rho_1 = \rho(\mathbf{X})$  is the mass density in the initial (unperturbed) state (configuration),  $\rho = \rho(\mathbf{x})$  is the mass density in the deformed state (configuration),  $J$  is the Jacobian of the transformation defined as the determinant of the transformations coefficients  $\alpha_{ij}$  (note:  $\alpha_{ij}$  is also called the Jacobian matrix), and  $O(u_{ij})$  are higher-order terms in  $u_{ij}$ . The Einstein summation of repeated indices is assumed in the present work.

The internal energy per unit mass  $U(\mathbf{x}, S)$  of the solid depends only on the relative positions of the particles comprising the solid and the entropy per unit mass  $S$ . Hence,  $U(\mathbf{x}, S)$  may be regarded as a function of the displacement gradients  $u_{ij}$ ,  $U(\mathbf{x}, S) = U(\mathbf{X}, u_{ij}, S)$ , or as a function of the Lagrangian strains  $\eta_{ij}$ ,  $U(\mathbf{x}, S) = U(\mathbf{X}, \eta_{ij}, S)$ , where the  $\eta_{ij}$  are defined by [18]

$$\eta_{ij} = \frac{1}{2}(\alpha_{ki}\alpha_{kj} - \delta_{ij}) = \frac{1}{2}(u_{ij} + u_{ji} + u_{ki}u_{kj}). \quad (3)$$

(Note that the Lagrangian strains are symmetric but the displacement gradients are generally not). Thus, we may expand the internal energy per unit mass either in Lagrangian strains or in displacement gradients as

$$\begin{aligned} \rho_1 U(\mathbf{x}, S) &= \rho_1 U(\mathbf{X}, \eta_{ij}, S) = \rho_1 U(\mathbf{X}, 0, S) + C_{ij}\eta_{ij} + \frac{1}{2}C_{ijkl}\eta_{ij}\eta_{kl} \\ &\quad + \frac{1}{3!}C_{ijklmn}\eta_{ij}\eta_{kl}\eta_{mn} + \cdots \\ &= \rho_1 U(\mathbf{X}, u_{ij}, S) = \rho_1 U(\mathbf{X}, 0, S) + A_{ij}u_{ij} + \frac{1}{2}A_{ijkl}u_{ij}u_{kl} \\ &\quad + \frac{1}{3!}A_{ijklmn}u_{ij}u_{kl}u_{mn} + \cdots, \end{aligned} \quad (4)$$

where  $A_{ij}$ ,  $A_{ijkl}$ , and  $A_{ijklmn}$ , respectively, are the first, second, and third-order Huang coefficients and  $C_{ij}$ ,  $C_{ijkl}$ , and  $C_{ijklmn}$ , respectively, are the first, second, and third-order Brugger elastic constants [10,27]. Substituting Eq. (3) in Eq. (4), expanding the Lagrangian strains in terms of the displacement gradients, and comparing the coefficients of like powers of the displacement gradients, we obtain the relations

$$A_{ij} = C_{ij} = T_{ij}(\mathbf{X}), \quad (5)$$

$$A_{ijkl} = T_{jl}(\mathbf{X})\delta_{ik} + C_{ijkl} \quad (6)$$

and

$$A_{ijklmn} = C_{jlmn}\delta_{ik} + C_{ijnl}\delta_{km} + C_{jnkl}\delta_{im} + C_{ijklmn}. \quad (7)$$

The first-order constants  $A_{ij} = C_{ij} = T_{ij}(\mathbf{X})$  are the initial stresses in the solid. We note from Eq. (4) that the Brugger constants are completely symmetric with respect to an exchange of index pairs and with respect to an exchange of indices within each pair, but Eqs. (5)–(7) show that the Huang coefficient do not generally obey such symmetries.

The stresses referred to the Eulerian (spatial) coordinates are the Cauchy stresses  $T_{ij}$ . The Cauchy stresses, evaluated in the present (perturbed) configuration  $\mathbf{x}$ , are given by [18,24,27]

Download English Version:

<https://daneshyari.com/en/article/10690542>

Download Persian Version:

<https://daneshyari.com/article/10690542>

[Daneshyari.com](https://daneshyari.com)