Ultrasound in Med. & Biol., Vol. 40, No. 4, pp. 695–703, 2014

0301-5629/$ - see front matter

ELSEVIER

http://dx.doi.org/10.1016/j.ultrasmedbio.2013.09.033

● *Original Contribution for the Festschrift in honor of Robert C. Waag*

COMPARISON OF FRACTIONAL WAVE EQUATIONS FOR POWER LAW ATTENUATION IN ULTRASOUND AND ELASTOGRAPHY

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(*Received* 9 *July* 2013; *revised* 27 *September* 2013; *in final form* 30 *September* 2013)

Abstract—A set of wave equations with fractional loss operators in time and space are analyzed. The fractional Szabo equation, the power law wave equation and the causal fractional Laplacian wave equation are all found to be low-frequency approximations of the fractional Kelvin-Voigt wave equation and the more general fractional Zener wave equation. The latter two equations are based on fractional constitutive equations, whereas the former wave equations have been derived from the desire to model power law attenuation in applications like medical ultrasound. This has consequences for use in modeling and simulation, especially for applications that do not satisfy the low-frequency approximation, such as shear wave elastography. In such applications, the wave equations based on constitutive equations are the viable ones. (E-mail: sverre@ifi.uio.no) © 2014 World Federation for Ultrasound in Medicine & Biology.

Key Words: **Fractional derivative, Elastography, Constitutive equations, Power law, Absorption, Lossy wave equation, Viscoelastic, Ultrasound.**

INTRODUCTION

Models for ultrasound attenuation based on relaxation losses have a long history (see, *e.g.*, Bhatia 1967; Markham et al. 1951). Waag and his co-workers also contributed with their work on multiple relaxation losses (Nachman et al. 1990). Similar, but much simpler relaxation models are used for attenuation in salt water and air. In salt water, the two important relaxation processes are due to boric acid and magnesium sulfate (Ainslie and McColm 1998), and in air, they are due to nitrogen and oxygen (Bass et al. 1995). In contrast, in medical ultrasound and elastography, attenuation for both compressional and shear waves often follows a power law that at first sight is very different from a relaxation model:

$$\alpha_k(\omega) = \alpha_0 \omega^y \tag{1}$$

Here, y usually is between 0 and 2, and ω is angular frequency. The subscript k is used to indicate that this is the imaginary part of the wavenumber, k, and to distinguish it from the order of the fractional derivative, α, which will be used later.

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In such media, the number of relaxation processes may in practice be uncountable, and it has not been possible to make models for attenuation that are so grounded in physical processes as for salt water and air. Nevertheless, the multiple relaxation model with a few processes can be used to model the power law attenuation, as Waag's group and several others have done (Tabei et al. 2003; Yang and Cleveland 2005) over a limited frequency range. But the relaxation parameters lose their clear physical meaning in this case and become just parameters of a mathematical model. Thus, on the one hand, there is power law attenuation, which is experimentally observed in many different complex materials, and on the other hand, physical models, exemplified by the multiple relaxation model.

In the last decade or so, advances have been made in providing wave equations that partially bridge this gap. They model power law attenuation through the use of fractional (*i.e.,* non-integer) derivatives. To varying degrees these equations are physics based, but especially for some viscoelastic polymers, there is a good physical foundation (Näsholm and Holm 2013). However, there is seldom as direct a connection as for the relaxation models for air and salt water. Nevertheless, they represent one step on the way to a deeper understanding of the underlying physics. They also provide alternative simulation models for wave propagation, often characterized by being parametrized with a small number of parameters.

For this article, the time fractional derivative is easiest to define in the frequency domain, where it is an extension of the Fourier transform of an nth-order derivative:

$$\mathscr{F}\left(\frac{d^{\alpha}u(t)}{dt^{\alpha}}\right)=(i\omega)^{\alpha}U(\omega) \tag{2}$$

The fractional derivative of arbitrary order can be understood as a generalization where the integer n is replaced with a real number α. The close connection between power laws and fractional derivatives is evident from this definition. The time domain equivalent involves a convolution integral implying that the fractional derivative is non-local and has a power law-shaped memory (Podlubny 1999).

Our purpose in this article is to analyze and compare several of the fractional wave equations, to understand their origins and to relate them to each other. The ultimate objective is to find wave equations that are physically viable. Another more practical objective is to determine which wave equations are best suited for simulation of medical ultrasound and shear wave elastography. One of the questions asked is: What is the most fundamental property to model with a wave equation? Is the objective only to model a power law attenuation, or is it more fundamental to model a medium with viscoelastic properties so that different power law attenuation laws are achieved in different frequency regions?

Because it is usually the low-frequency region that is of interest, many wave equations model only this region. But there are applications where the high-frequency region is important also; one particularly important one is in shear wave elastography. Even when the interest is only in the low-frequency solution, such as for ultrasound imaging, then for the solution to be physically viable, it is often an advantage that wave equations give correct solutions beyond this region. The key to obtaining such results is the viscoelastic constitutive equation. This article, therefore, builds a case for the viewpoint that it is the viscoelastic constitutive equation that is the more fundamental and physical property, rather than the power law characteristics.

It should be noted that here, low frequencies means low in comparison to the value of a time constant (τ_σ in, *e.g.*, eqn (4)). Thus, for example, for compressional waves in medical ultrasound, low frequencies stretch well beyond the usual range and up into the hundreds of megahertz or higher. For such an application, the power law has an exponent, y, between 1 and 2. On the other hand, it turns out that for shear waves in the human body in elastography, the low-frequency limit is $<$10 Hz (Holm and Sinkus 2010). Thus, elastography operates above the low-frequency limit with y less than unity.

Physically viable solutions

There are some criteria that need to be satisfied by the solutions of wave equations for them to be physically viable. The first is that of causality: No output is allowed before there is an input.

The second property is that the loss function cannot rise arbitrarily fast and has to satisfy

$$\lim_{\omega\to\infty}\alpha_k(\omega)/\omega=0 \tag{3}$$

This means that the exponent, y, has to be less or equal to one in the high-frequency limit. This condition was first given by Weaver and Pao (1981) and more rigorously by Hanyga (2013). It follows from causality, passivity and linearity and is a consequence of the property that viscoelasticity can be modeled as a sum of many spring-dashpot mechanical models with realistic, that is, positive, constants (Beris and Edwards 1993).

FRACTIONAL WAVE EQUATIONS

We will give wave equations for plane waves in a homogenous medium. Figure 1 illustrates how the wave is modified by transmission through a slab of thickness z. The wave number is, in general, complex and given by $k=\beta_k - i\alpha_k$, where the real part describes dispersion and the imaginary part gives the attenuation. Note that many papers use the opposite sign for the exponent in the traveling wave. This will result in somewhat different wave equations, as illustrated in Appendix A, which has been included to facilitate comparisons between articles using different sign conventions.

Fractional time loss operator, type 1

One of the oldest fractional wave equations is due to Caputo (1967). In the terminology of Wismer (2006), it is

$$\nabla^2 u-\frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2}+\tau_\sigma^{y-1}\frac{\partial^{y-1}}{\partial t^{y-1}}\nabla^2 u=0, \tag{4}$$

where u is the displacement, c_0 is the phase velocity at zero frequency, $\alpha_0=\tau_\sigma/(2c_0)$ in eqn (1) and the condition

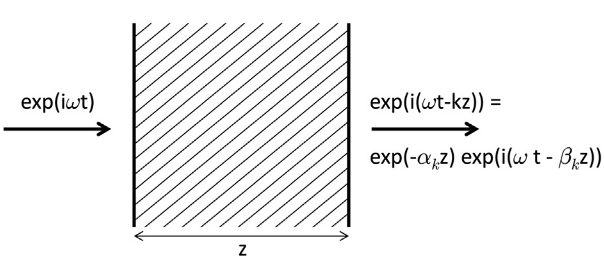

Fig. 1. A slab of thickness z with a plane wave traveling from left to right.