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Structural stability of rubble-pile asteroids

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ABSTRACT

Granular aggregates, like fluids, do not admit all manners of shapes and rotation rates. It is hoped that an analysis of a suspected granular asteroid's equilibrium shape and its structural stability will help confirm its rubble-pile nature, and, perhaps, even constrain the asteroid's material parameters. Equilibrium shapes have been analyzed in the past by several investigators (Holsapple, K.A. [2001]. Icarus 154, 432–448; Harris, A.W., Fahnestock, E.G., Pravec, P. [2009]. Icarus 199, 310–318; Sharma, I., Jenkins, J.T., Burns, J.A. [2009]. Icarus 200, 304–322). Here, we extend the classical Lagrange–Dirichlet stability theorem to the case of self-gravitating granular aggregates. This stability test is then applied to probe the stability of several near-Earth asteroids, and explore the influence of material parameters such as internal friction angle and plastic bulk modulus. Finally, we consider their structural stability to close planetary encounters. We find that it is possible for asteroids to be stable to small perturbations, but unstable to strong and/or extended perturbations as experienced during close flybys. Conversely, assuming stability in certain situations, it is possible to estimate material properties of some asteroids like, for example, 1943 Anteros.

1. Introduction

We are interested in testing the structural stability of freely-rotating rubble-pile asteroids. The equilibrium shapes of these objects have been analyzed previously by Holsapple (2001) via limit analysis, by Harris et al. (2009) who bound the surface's maximum slope by the local angle of repose, and by Sharma et al. (2009) utilizing volume-averaging. At the same time, Richardson et al. (2005) explored the equilibrium shapes of a collective of same-sized smooth rigid spheres via a hard-particle discrete-element simulation, while more recently Sanchez and Scheeres (2011, 2012) have utilized soft-particle discrete element simulations to investigate the equilibria and dynamics of granular aggregates in space.

Much work is available in the stability of rotating fluid ellipsoids subjected to gravitational and tidal forces; see, e.g., Chandrasekhar (1969), Jeans (1961) and Lyttleton (1953). In contrast, except for the recent work of Holsapple (2004), the stability of spinning ellipsoidal granular aggregates has not been explored. As we will see, typical stability tests employed for fluids do not carry over in a straightforward manner, if at all, to the case of rubble-piles on account of them being most conveniently modeled as *non-smooth* materials; cf. Section 3. A stability test for rotating non-smooth complex fluids was recently developed by Sharma (2012), and we extend it to the case of granular aggregates. We will utilize our stability test to investigate several near-Earth asteroids. We also compare our approach with that of Holsapple (2004) at Section 7.3's end. Finally, we extend our analysis of local stability under infinitesimal perturbations to include finite disturbances via an approximate higher-order stability analysis.

In our analysis, we will model rubble-pile asteroids as homogeneous isotropic self-gravitating rigid-plastic ellipsoids, and, furthermore, restrict ourselves to deformations that retain the body's ellipsoidal shape, i.e., *homogeneous deformations*. Sharma et al. (2009), henceforth Paper I, derives relevant dynamical equations. For reasons discussed later, the stability of rotating bodies is best tested from within a rotating coordinate system. Thus, we first re-derive the governing dynamical equations below, separating out the effects of an underlying rotating coordinate system.

2. Homogeneous dynamics in a rotating frame

Sharma (2012), henceforth Paper II, derives equations for a homogeneously deforming ellipsoid in a rotating frame from first principles. Here, for reasons of completeness and continuity, we adapt the derivation of Paper I done in a fixed coordinate system to a rotating frame.

We recall that during homogeneous deformation, the material velocity

$$\mathring{\mathrm{x}} = \boldsymbol{L}_F \cdot \mathrm{x}, \tag{1}$$

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with x locating a material point, $(\overset{\circ}{\ })$ denoting the time derivative in a fixed frame[1] and $\boldsymbol{L}_F$ being the motion's velocity gradient that is spatially homogeneous but possibly time-varying; here the subscript 'F' indicates that the velocity gradient is with respect to a fixed frame. Equations governing a homogeneously deforming ellipsoid free from surface forces but experiencing the body force[2] b are (Sharma et al., 2009):

$$\left(\overset{\circ}{\boldsymbol{L}}_F + \boldsymbol{L}_F^2\right) \cdot \boldsymbol{I} = -\bar{\boldsymbol{\sigma}} V + \boldsymbol{M}^T, \quad \text{and} \tag{2a}$$

$$\overset{\circ}{\boldsymbol{I}} = \boldsymbol{L}_F \cdot \boldsymbol{I} + \boldsymbol{I} \cdot \boldsymbol{L}_F^T, \tag{2b}$$

where $\bar{\boldsymbol{\sigma}}$ is the volume-averaged stress tensor,

$$\boldsymbol{I} = \int_V \rho \mathrm{x} \otimes \mathrm{x} \, dV, \quad \text{and} \tag{3a}$$

$$\boldsymbol{M} = \int_V \rho \mathrm{x} \otimes \mathrm{b} \, dV, \tag{3b}$$

being, respectively, the ellipsoid's inertia tensor and external moment tensor, ρ and V are the ellipsoid's density and volume, respectively, and the *tensor product* $(\otimes)$ for any two vectors a and b is defined in indical notation by $(\mathrm{a} \otimes \mathrm{b})_{ij} = a_i b_j$. The first equation above follows $\boldsymbol{L}_F$'s evolution by balancing inertial forces, internal stresses and self-gravitation, while the second describes the changing inertia tensor. The inertia tensor $\boldsymbol{I}$ should be contrasted with Euler's moment of inertia tensor frequently employed in dynamics; cf. (23). Finally, we refer the reader to the Appendix of Sharma (2009) for a short summary of relevant tensor algebra.

We now adapt the above formulae to a coordinate system $\mathcal{O}$ rotating at a possibly time-varying rate $\boldsymbol{\omega}(t)$. It is helpful to introduce the anti-symmetric tensor $\boldsymbol{\Omega}(t)$ that satisfies

$$\boldsymbol{\omega} \times \mathrm{x} = \boldsymbol{\Omega} \cdot \mathrm{x}; \tag{4}$$

thus, $\boldsymbol{\omega}$ is $\boldsymbol{\Omega}$'s associated *axial vector*. We employ rotation rate vectors, such as $\boldsymbol{\omega}$, and their corresponding *angular-velocity tensors*, like $\boldsymbol{\Omega}$, interchangeably. From, say, Lubarda (1999, p. 41), the rates of change in $\mathcal{O}$ of a vector a and a tensor $\boldsymbol{B}$ may be related to their rates of change in a fixed frame by, respectively,

$$\overset{\circ}{\mathrm{a}} = \dot{\mathrm{a}} + \boldsymbol{\Omega} \cdot \mathrm{a} \quad \text{and} \quad \overset{\circ}{\boldsymbol{B}} = \dot{\boldsymbol{B}} + \boldsymbol{\Omega} \cdot \boldsymbol{B} - \boldsymbol{B} \cdot \boldsymbol{\Omega},$$

where now, and henceforth, $(\dot{\ })$ indicates a time derivative in the rotating frame $\mathcal{O}$. Thus, if $\hat{\mathrm{e}}_i$ are unit vectors defining $\mathcal{O}$, then $\dot{\mathrm{a}} = \overset{\circ}{a}_i \hat{\mathrm{e}}_i$ and $\dot{\boldsymbol{B}} = \overset{\circ}{B}_{ij} \hat{\mathrm{e}}_i \otimes \hat{\mathrm{e}}_j$. Viewing the homogeneously deforming ellipsoid from within $\mathcal{O}$, we may now rewrite (1) as

$$\mathrm{v} = \dot{\mathrm{x}} = \boldsymbol{L} \cdot \mathrm{x}, \tag{5}$$

where v is the relative velocity of a material point in $\mathcal{O}$ and

$$\boldsymbol{L} = \boldsymbol{L}_F - \boldsymbol{\Omega}, \tag{6}$$

is the velocity gradient observed in the rotating frame $\mathcal{O}$. Similarly, the set (2) becomes

$$(\dot{\boldsymbol{L}} + \boldsymbol{L}^2) \cdot \boldsymbol{I} = -\bar{\boldsymbol{\sigma}} V + \boldsymbol{M}^T - (\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega}^2 + 2\boldsymbol{\Omega} \cdot \boldsymbol{L}) \cdot \boldsymbol{I}, \quad \text{and} \tag{7a}$$

$$\dot{\boldsymbol{I}} = \boldsymbol{L} \cdot \boldsymbol{I} + \boldsymbol{I} \cdot \boldsymbol{L}^T. \tag{7b}$$

We note that the stress and moment tensors remain unaffected by rigid body rotation, as does the form for $\boldsymbol{I}$'s rate of change. In the first equation, the bracketed three terms on the right-hand side stem from, respectively, angular, centripetal and Coriolis' accelerations, and act in the rotating frame as external moment tensors. The two equations above follow an ellipsoid's motion as it deforms homogeneously relative to the frame $\mathcal{O}$ according to (5).

The above two equations contain three unknown fields, viz., $\boldsymbol{L}, \bar{\boldsymbol{\sigma}}$ and $\boldsymbol{I}$. Thus, a closure equation is required. This is provided by introducing a constitutive law. Given our interest in rubble-pile asteroids, we will next introduce a rheology that describes these granular aggregates.

[1] We slightly modify Paper I's notation for future convenience.

[2] Force per unit mass.

3. Rheology

In the past, Sharma (2004, 2009, 2010) and Sharma et al. (2005, 2009) have modeled a rubble-pile's constitutive response by that of a rigid perfectly-plastic cohesionless material obeying a Drucker–Prager yield criterion and an appropriate *flow rule* that governed the material's behavior post-yield. Paper I chose for simplicity a *non-associative* flow rule that preserved volume during plastic flow. Here we consider an *associative* flow rule; non-associative rigid-plastic materials are typically trivially *secularly*[3] unstable, as we show in a later section.

We first quickly introduce the Drucker–Prager yield criterion whose smoothness makes it amenable to three-dimensional dynamic problems. To formulate this rule, we define the pressure

$$p = -\frac{1}{3} \mathrm{tr} \boldsymbol{\sigma}, \tag{8}$$

and the deviatoric stress

$$\boldsymbol{s} = \boldsymbol{\sigma} + p\boldsymbol{1}. \tag{9}$$

The Drucker–Prager condition may now be written as

$$|\boldsymbol{s}|^2 \leqslant k^2 p^2, \tag{10}$$

where $|\boldsymbol{s}|$ is $\boldsymbol{s}$' magnitude given by

$$|\boldsymbol{s}|^2 = s_{ij} s_{ij},$$

utilizing the summation convention, and

$$k = \frac{2\sqrt{6} \sin \phi_F}{3 - \sin \phi_F}, \tag{11}$$

in terms of the granular aggregate's internal friction angle ϕ_F. Employing the principal stresses σ_i, we have

$$|\boldsymbol{s}| = \frac{1}{3}\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}$$
$$= \frac{2}{3}(\tau_1^2 + \tau_2^2 + \tau_3^2), \tag{12}$$

where $\tau_i = (\sigma_j - \sigma_k)/2, \ i \neq j \neq k$ are the principal shear stresses. Therefore, $|\boldsymbol{s}|$ is a measure of the 'total' shear stress, and, consequently, the yield criterion (10) limits the shear stress in terms of the pressure and the internal friction angle. This internal friction models the ability of an aggregate to support shear stresses, and is traced to both the usual *interfacial* friction due to particle interaction, as well as a *geometric* friction due to interlocking and rearrangement of finite-sized constituents. The latter is generally dominant in dense aggregates, but decreases with lowering density. Similarly, ϕ_F is greatly affected by confining pressure, with grain crushing lowering ϕ_F when the confining pressure is 1 MPa or beyond. Typically dense soils at these confining pressures display ϕ_F between 30° and 40°. We discuss soil behavior in more detail in Section 7.1. In passing, it is worth mentioning the alternate Mohr–Coulomb yield criterion wherein relation (10) is phrased in terms of the greatest shear stress rather than $|\boldsymbol{s}|$; see Chen and Han (1988, Section 2.3.3, p. 88). This yield criterion was utilized by Holsapple (2001, 2004). Sharma et al. (2005) and later Sharma et al. (2009) showed that utilizing Drucker–Prager yield surface allowed for a better match with simulations of Richardson et al. (2005). The recent simulations of Sanchez and Scheeres (2012) confirm this latter prediction.

[3] Systems found stable by the energy criterion are secularly stable, cf. Section 5.2.