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A graphical interpretation of the electrical conductivity tensor

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Abstract

Electrical conductivity plays a central role in many areas of space science. However, texts and research articles sometimes misquote the conductivity tensor. This highlights the need for authors and researchers to verify the accuracy of this tensor for their chosen coordinate system. This paper presents a new graphical method to analyze the correctness of the conductivity tensor, applicable to any coordinate system. This method also illuminates the physical meaning of the terms in the conductivity tensor, which is often obscured by standard mathematical derivations.
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1. Introduction

Electrical conductivity is a central concept in space science. It determines how driving forces, such as electric fields and thermospheric winds, couple to plasma motions and the resulting electric currents. The tensor of electrical conductivity, $\boldsymbol{\sigma}$, finds application in all areas of ionospheric electrodynamics, and at all latitudes. For example, near the magnetic equator, the enhanced current of the equatorial electrojet is explained by the existence of a large perpendicular conductivity, known as the Cowling conductivity (often labeled as σ_3). The expression for σ_3 is derived from the 2×2 representation of $\boldsymbol{\sigma}$, in which the vertical current, J_z, is presumed to be zero. Other common low- and mid-latitude applications of $\boldsymbol{\sigma}$ include the *E*- and *F*-region dynamos driven by neutral winds. At high latitudes, gradients in Hall and Pedersen conductivities—along with the convergence or divergence of the convection electric fields—explain why *E*-region currents (such as the auroral electrojet) cannot close in the plane perpendicular to the magnetic field, and thus field-aligned currents must exist in order to satisfy the steady-state relation $\nabla \bullet \mathbf{J} = 0$. In addition, Kamide (1988) reviews—and expands upon—several studies to explain high-latitude storm-time current systems in terms of a "Cowling Channel." As a final example of the ubiquitous nature of conductivity in ionospheric science, knowledge of the relationship between the current density and ionospheric electric fields (which $\boldsymbol{\sigma}$ provides) is essential to determine the morphology of Joule heating, which in turn impacts ionospheric thermal structure and chemical reaction rates.

Naturally, expressions for $\boldsymbol{\sigma}$ appear throughout many classic textbooks (Rishbeth and Garriott, 1969 (hereafter referred to as "R&G"); Whitten and Poppoff, 1971; Kelley, 1989) and in the journal literature, such as the landmark equatorial electrojet review by Forbes (1981). Unfortunately, the $\boldsymbol{\sigma}$ tensor is sometimes misstated in the literature, or at the very least authors may fail to give an adequate description of their chosen coordinate system. This "disconnect" can be quite

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confusing for researchers and teachers alike. Adding to the difficulty is the fact that conductivity discussions are purely mathematical throughout the literature; I could not find any graphical explanations of the elements of the $\boldsymbol{\sigma}$ tensor, analogous to the air drag and ion drag treatments of R&G and Schunk and Nagy (2000). It seems clear that such graphical aids are key to exposing the physical meaning of $\boldsymbol{\sigma}$ (relative to a given coordinate system); this paper provides a new graphical tool to explain the elements of the electrical conductivity tensor.

2. Derivation of the $\boldsymbol{\sigma}$ tensor

It is useful to review the basic derivation of the conductivity tensor. For this we follow the discussions in chapter four of R&G. They begin with the steady-state momentum equation for species j (either ions or electrons), ignoring gravity, pressure gradient, and stress forces, as well as momentum transfer due to Coulomb collisions:

$$\mp e(\mathbf{E} + \mathbf{V}_j \times \mathbf{B}) = m_j \sum_n v_{jn}(\mathbf{U}_n - \mathbf{V}_j), \tag{1}$$

where e is the charge magnitude of an ion or electron (the upper sign is for ions, lower for electrons), m_j is the mass, v_{jn} is the momentum transfer collision frequency for species j due to collisions with the neutral gas, $\mathbf{V}_j$ is the bulk flow velocity of species j, $\mathbf{U}_n$ is the neutral wind velocity, and the sum is taken over all neutral species. The steady-state assumption implies that Eq. (1) is valid on time scales long compared to both v_{jn}^{-1} and ω_j^{-1}, where ω_j is the gyrofrequency of species j.

At this point, R&G assume the plasma contains one ion species, one neutral species, and electrons. Further, they take $\mathbf{U}_n = 0$; this is equivalent to the statement that the electric field $\mathbf{E}$ is valid in the neutral wind rest frame. Eq. (1) can be solved for $\mathbf{V}_i$ and $\mathbf{V}_e$, and then the electric current density is found from

$$\mathbf{J} = Ne(\mathbf{V}_i - \mathbf{V}_e) = \boldsymbol{\sigma} \bullet E, \tag{2}$$

where N is the electron density. If the magnetic field is aligned with the z-axis of our coordinate system, the conductivity tensor in Eq. (2) is expressed as

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & -\sigma_2 & 0 \\ \sigma_2 & \sigma_1 & 0 \\ 0 & 0 & \sigma_0 \end{pmatrix}, \tag{3}$$

where σ_0 is the longitudinal conductivity, σ_1 is the Pedersen conductivity, and σ_2 is the Hall conductivity, as given in R&G equation (418). The interpretation of Eq. (3) is quite simple, but in preparation for the next section, we will graphically analyze the current density $\mathbf{J}$ due to an electric field in the $+x$ direction. This situation is depicted in Fig. 1. In this case, the current density is

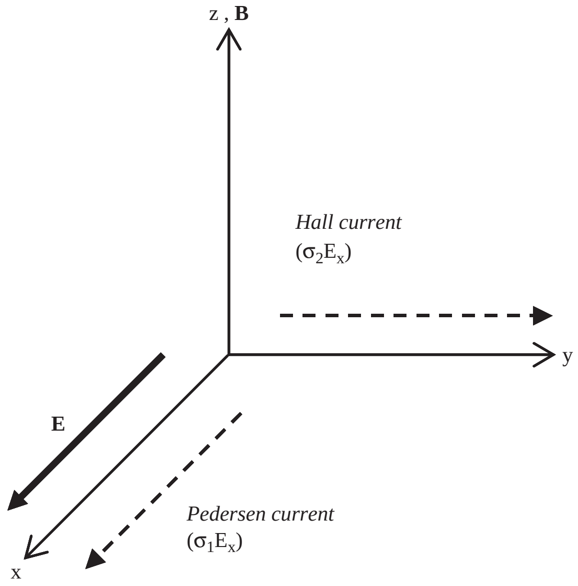

Fig. 1. Pedersen and Hall currents resulting from an electric field $\mathbf{E}$ in the $+x$-direction. The magnetic field is directed along the $+z$-axis.

$$\mathbf{J} = \boldsymbol{\sigma} \bullet \begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_1 E_x \\ \sigma_2 E_x \\ 0 \end{pmatrix}. \tag{4}$$

This equates to a Pedersen current in the $+x$ direction and a Hall current in the $+y$ direction, as shown in Fig. 1. The Hall current is in the $\mathbf{B} \times \mathbf{E}$ direction because, at any altitude, the electron Hall mobility is greater than or equal to the ion Hall mobility. Also note that Eq. (4) is a general result, applicable at any altitude. Of course, σ_1 and σ_2 become small outside the dynamo region.

Now in ionospheric physics it is often convenient to perform a similarity transformation on Eq. (3) to convert to a coordinate system in which the z-axis is vertical; Fig. 2 illustrates this process. We begin with a coordinate system in which the y-axis is directed toward *geomagnetic east*, the z-axis is along $\mathbf{B}$, and the x-axis orientation satisfies the right-hand rule. The dip angle, I, is positive in the figure. To obtain the desired coordinate orientation, we rotate the coordinate system clockwise about the y-axis by an angle $I + \pi/2$. As Fig. 2 shows, the resulting coordinate system will have its x-axis parallel to the ground, pointing toward geomagnetic *south*; a compass needle would indicate this direction as *northward*. The y-axis will still point toward geomagnetic *east* (i.e., in the general direction of *geographic west*), and the z-axis will point toward the local zenith. The appropriate rotation matrix to produce