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Multi-loop zeta function regularization and spectral cutoff in curved spacetime

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Abstract

We emphasize the close relationship between zeta function methods and arbitrary spectral cutoff regularizations in curved spacetime. This yields, on the one hand, a physically sound and mathematically rigorous justification of the standard zeta function regularization at one loop and, on the other hand, a natural generalization of this method to higher loops. In particular, to any Feynman diagram is associated a generalized meromorphic zeta function. For the one-loop vacuum diagram, it is directly related to the usual spectral zeta function. To any loop order, the renormalized amplitudes can be read off from the pole structure of the generalized zeta functions. We focus on scalar field theories and illustrate the general formalism by explicit calculations at one-loop and two-loop orders, including a two-loop evaluation of the conformal anomaly.

Keywords: Zeta function regularization; Quantum field theory in curved spacetime; Higher loops in curved spacetime; Spectral cutoff regularization; Conformal anomaly

1. General presentation

1.1. Introduction

Quantum field theory in curved spacetime is a mature area of research with many outstanding applications, including particle creation in time-dependent background and black hole

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evaporation (see e.g. [1] and references therein). Interesting applications to the calculation of the leading quantum corrections to the area law for the black hole entropy have also appeared recently [2]. However, the subject has been almost entirely focusing on free fields or, equivalently, on the one-loop vacuum energy. One of the difficulties to compute at higher loops is to define an appropriate reparameterization invariant regularization scheme. In principle, one may use dimensional regularization, but this scheme it is not very natural in curved spacetime because there is no canonical way to generalize a general d-dimensional spacetime manifold $\mathscr{M}_d$ to arbitrary $d+\epsilon$ dimensions. A much preferred and powerful regularization method is the zeta function scheme [3]. This approach is very elegant and manifestly reparameterization invariant. However, it is only defined at the one-loop level. The main goal of the present work is to show that zeta function methods are also very natural at higher-loop order, by highlighting a close relationship between the zeta function scheme and the general physical spectral cutoff regularization.

1.2. On the zeta function regularization

As a simple illustration of the zeta function method, let us consider a massless scalar field on a two-dimensional spacetime of the form $\mathbb{R}\times\mathrm{S}^1$, the length of the circle being a. Its momentum is quantized in units of $2\pi/a$ and the vacuum energy is formally given by an infinite sum,

$$E=\frac{2\pi}{a}\sum_{n>0}n. \tag{1.1}$$

The zeta function prescription amounts to replacing the above ill-defined sum by the analytic continuation of the Riemann ζ function

$$\zeta_{\mathrm{R}}(s)=\sum_{n>0}\frac{1}{n^s} \tag{1.2}$$

at the physically relevant value $s=-1$. ζ_{R} is a meromorphic function with a single pole at $s=1$ with unit residue and $\zeta_{\mathrm{R}}(-1)=-\frac{1}{12}$. Hence

$$E_\zeta=\frac{2\pi}{a}\zeta_{\mathrm{R}}(-1)=-\frac{\pi}{6a}. \tag{1.3}$$

Much more generally, a typical one-loop calculation in curved spacetime involves the computation of a Gaussian path integral which yields the determinant of some wave operator D. For example, in the case of a scalar field on a Euclidean Riemannian manifold endowed with a metric g,

$$D=\Delta+m^2+\xi R, \tag{1.4}$$

where Δ is the positive Laplace–Beltrami operator, m the mass parameter, ξ an arbitrary dimensionless constant and R the Ricci scalar. If we denote the eigenvalues of D by λ_r, the determinant of D is formally given by an infinite product

$$\det D=\prod_r\lambda_r. \tag{1.5}$$

In the zeta function scheme, this infinite product is defined by introducing the spectral zeta function associated with the wave operator D,

$$\zeta_D(s)=\sum_r\frac{1}{\lambda_r^s}. \tag{1.6}$$