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A generalized Beraha conjecture for non-planar graphs

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Abstract

We study the partition function $Z_{G(nk,k)}(Q, v)$ of the Q-state Potts model on the family of (non-planar) generalized Petersen graphs $G(nk, k)$. We study its zeros in the plane (Q, v) for $1 \leqslant k \leqslant 7$. We also consider two specializations of $Z_{G(nk,k)}$, namely the chromatic polynomial $P_{G(nk,k)}(Q)$ (corresponding to $v = -1$), and the flow polynomial $\Phi_{G(nk,k)}(Q)$ (corresponding to $v = -Q$). In these two cases, we study their zeros in the complex Q-plane for $1 \leqslant k \leqslant 7$. We pay special attention to the accumulation loci of the corresponding zeros when $n \to \infty$. We observe that the Berker–Kadanoff phase that is present in two-dimensional Potts models, also exists for non-planar recursive graphs. Their qualitative features are the same; but the main difference is that the role played by the Beraha numbers for planar graphs is now played by the non-negative integers for non-planar graphs. At these integer values of Q, there are massive eigenvalue cancellations, in the same way as the eigenvalue cancellations that happen at the Beraha numbers for planar graphs.

Keywords: Potts model; Non-planar graphs; Beraha conjecture; Generalized Petersen graphs; Transfer matrix; Berker–Kadanoff phase

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1. Introduction

The two-dimensional (2D) Q-state Potts model [41,58] is one of the most studied models in Statistical Mechanics. Despite many efforts over more than 60 years, its *exact* free energy and phase diagram are still unknown. The ferromagnetic regime of the Potts model is the best understood case: exact (albeit not always rigorous) results have been obtained for the ferromagnetic–paramagnetic phase transition temperature $T_c(Q)$ for several regular lattices, the order of the transition (continuous for $0 \leqslant Q \leqslant 4$, and first order for $Q > 4$), the phase diagram, and the characterization in terms of conformal field theory (CFT) of the corresponding universality classes. (See e.g., Ref. [3].)

The antiferromagnetic (AF) regime is less understood. This is partly because, in contrast with the ferromagnetic regime, universality cannot be expected to hold in general. Investigations must therefore proceed on a case-by-case basis. For instance, the free energy is known exactly along some curves of the phase diagram (Q, T) (where T is the temperature), for certain regular 2D lattices [3,4]. One of these curves belongs to the ferromagnetic regime, and it can be identified with the ferromagnetic–paramagnetic phase-transition curve. It might be tempting to infer the very existence of a phase transition from this (partial) solubility of the model along that curve. One well-known example of the invalidity of such an inference is provided by the zero-temperature limit of the triangular-lattice Q-state Potts antiferromagnet [5,6]. Although its free energy is exactly known for all values of $Q \in \mathbb{R}$,[1] the system is known to be critical only in the interval $Q \in [0, 4]$, and disordered for $Q \in (-\infty, 0) \cup (4, \infty)$.

In three dimensions (3D) there are no known exact results for the Q-state Potts model. Most numerical results come from series expansions and Monte Carlo simulations: see e.g., Refs. [23, 37, and references therein]. In the ferromagnetic regime, we expect a critical curve, which is second order for $Q = 2$, and first-order for $Q = 3$. There was an important controversy in the late 80s and early 90s about the precise nature of the $Q = 3$ transition because of its relation with QCD: the four-dimensional SU(3) lattice gauge theory should be in the same universality class as the 3D ferromagnetic 3-state Potts model [37].

The Q-state Potts model at temperature T can be defined on any (undirected) finite graph $G = (V, E)$ with vertex set V and edge set E. On each vertex $i \in V$, we place a spin that can take Q distinct values: $\sigma_i \in \{1, 2, \ldots, Q\}$. These spins interact through the Hamiltonian [41]

$$\mathcal{H}(\{\sigma\}) = -J \sum_{\langle ij \rangle \in E} \delta_{\sigma_i \sigma_j}, \tag{1.1}$$

where δ_{ij} is the usual Kronecker delta, and J is a *real* coupling constant that is proportional to $1/T$. The partition function is defined as usual as:

$$Z_G(Q, v) = \sum_{\{\sigma\}} e^{-\mathcal{H}}. \tag{1.2}$$

Notice that initially, Q is a positive integer $Q \geqslant 2$, and J is a real number. The ferromagnetic (resp. antiferromagnetic) regime corresponds to $J \geqslant 0$ (resp. $J \leqslant 0$).

Fortuin and Kasteleyn [22] have shown that the partition function (1.2) can be rewritten as

$$Z_G(Q, v) = \sum_{E' \subseteq E} v^{|E'|} Q^{k(E')}, \tag{1.3}$$

[1] The Q-state Potts model can be defined for non-integer values of Q using the Fortuin–Kasteleyn representation explained below [cf. (1.3)].