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Classical geometry from the quantum Liouville theory

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Abstract

Zamolodchikov's recursion relations are used to analyze the existence and approximations to the classical conformal block in the case of four parabolic weights. Strong numerical evidence is found that the saddle point momenta arising in the classical limit of the DOZZ quantum Liouville theory are simply related to the geodesic length functions of the hyperbolic geometry on the 4-punctured Riemann sphere. Such relation provides new powerful methods for both numerical and analytical calculations of these functions. The consistency conditions for the factorization of the 4-point classical Liouville action in different channels are numerically verified. The factorization yields efficient numerical methods to calculate the 4-point classical action and, by the Polyakov conjecture, the accessory parameters of the Fuchsian uniformization of the 4-punctured sphere.

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1. Introduction

A few years ago a considerable progress in the Liouville theory has been achieved [1]. The solution to the quantum theory based on the structure constants proposed by Otto and Dorn [2] and by A. and Al.B. Zamolodchikov [3] was completed by Ponsot and

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Teschner [4–6] and by Teschner [1,7]. Along with the techniques of calculating conformal blocks developed by Al.B. Zamolodchikov [8–10] the DOZZ theory provides explicit formulae for quantum correlators.

On the other hand there exists so-called geometric approach originally proposed by Polyakov [11] and further developed by Takhtajan [12–15] (see also [16–19]). In contrast to the operator formulation of the DOZZ theory the correlators of primary fields with elliptic and parabolic weights are expressed in terms path integral over conformal class of Riemannian metrics with prescribed singularities at the punctures. The underlying structure of this formulation is the classical hyperbolic geometry of the Riemann surface.

Although the relation between these formulations is not yet completely understood [20–22] it is commonly believed that the quasiclassical limit of the DOZZ theory exists and is correctly described by the classical Liouville action of the geometric approach. This is, for instance, justified by explicit calculation of the classical limit of the DOZZ structure constants and the classical Liouville action for the Riemann sphere with three punctures [3,23].

Some of the predictions derived from the path integral representation of the geometric approach can be rigorously proved and lead to deep geometrical results. This can be seen as an additional support for the correctness of the picture the geometric formulation provides for the semiclassical limit of the DOZZ theory. One of the results of this type is the so-called Polyakov conjecture obtained as a classical limit of the Ward identity [24–29]. It states that the classical Liouville action is a generating function for the accessory parameters of the Fuchsian uniformization of the punctured sphere yielding an essentially new insight into this classical long standing problem. Its usefulness for solving the uniformization is however restricted by our ability to calculate the classical Liouville action for more than three singularities.

The existence of the semiclassical limit of the Liouville correlation function with the projection on one intermediate conformal family implies a semiclassical limit of the BPZ quantum conformal block [30] with heavy weights $\Delta = Q^2\delta$, $\Delta_i = Q^2\delta_i$, with $\delta, \delta_i = \mathcal{O}(1)$ in the following form:

$$\mathcal{F}_{1+6Q^2,\Delta}\begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}(x) \sim \exp\left\{ Q^2 f_\delta \begin{bmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{bmatrix}(x) \right\}. \tag{1.1}$$

The function $f_\delta\left[\begin{smallmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{smallmatrix}\right](x)$ is called the classical conformal block [3] or (a bit confusing) the "classical action" [9,10].

The existence of the semiclassical limit (1.1) was first postulated in [9,10] where it was pointed out that the classical block is related to a certain monodromy problem of a null vector decoupling equation in a similar way the classical Liouville action is related to the Fuchsian uniformization. This relation was further used to derive the $\Delta \to \infty$ limit of the conformal block and its expansion in powers of the q variable.

The 4-point function of the DOZZ theory can be defined as an integral of s-channel conformal blocks and the DOZZ couplings over the continuous spectrum of the theory. In the semiclassical limit the integrand can be expressed in terms of the 3-point classical Liouville action and the classical block, and the integral itself is dominated by the saddle