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Superconformal Ward identities and their solution

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Abstract

Superconformal Ward identities are derived for the four point functions of chiral primary BPS operators for $\mathcal{N} = 2, 4$ superconformal symmetry in four dimensions. Manipulations of arbitrary tensorial fields are simplified by introducing a null vector so that the four point functions depend on two internal R-symmetry invariants as well as two conformal invariants. The solutions of these identities are interpreted in terms of the operator product expansion and are shown to accommodate long supermultiplets with free scale dimensions and also short and semi-short multiplets with protected dimensions. The decomposition into R-symmetry representations is achieved by an expansion in terms of two variable harmonic polynomials which can be expressed also in terms of Legendre polynomials. Crossing symmetry conditions on the four point functions are also discussed.

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1. Introduction

Since the discovery of the AdS/CFT correspondence there has been a huge resurgence of interest in superconformal theories in four dimensions, for a review see [1]. In particular for the $\mathcal{N} = 4$ superconformal $SU(N)$ gauge theory, to which the AdS/CFT correspondence

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is most directly applicable, many new and exciting results have been obtained. Much has been discovered concerning the spectrum of operators and their scale dimensions both in the large N limit, through the supergravity approximation to the AdS/CFT correspondence, and also perturbatively as an expansion in the coupling g.

Of the operators present in the theory the simplest are the chiral primary operators belonging to $SU(4)_R$ R-symmetry representations with Dynkin labels $[0, p, 0]$. These are represented by symmetric traceless rank p tensors formed by gauge invariant traces of the elementary scalar fields and satisfy BPS like constraints so that they belong to short supermultiplets of the superconformal group $PSU(2, 2|4)$. They are therefore protected against renormalisation effects and have scale dimension $\Delta = p$. Their three point functions have been fully analysed in [2] and perturbative corrections shown to vanish in [3,4] and, using harmonic superspace, in [5]. For the case of $p = 2$, when the supermultiplet contains the energy–momentum tensor, the four point functions have been found both perturbatively [6] and in the large N limit [7]. Such results have also been extended more recently to chiral primary operators with $p = 3, 4$ [8–10]. The explicit results for the four point correlation functions has then allowed an analysis of those operators which contribute to the operator product expansion for two chiral primary operators [11–17].

To take the analysis of the operator product expansion of correlation functions beyond the lowest scale dimension operators for each $SU(4)_R$ representation it is necessary to have an explicit form for the conformal partial waves which give the contribution of a quasi-primary operator of arbitrary scale dimension and spin and all its conformal descendants to conformally covariant four point functions. In four dimensions a simple expression was found in [18]. In addition since all operators in a superconformal multiplet must have the same anomalous dimensions it is desirable to have a procedure for analysing the operator product expansion for each supermultiplet as a single contribution. This depends on a solution of all superconformal Ward identities since this should allow all possible operator product expansion contributions to be found in a form compatible with the superconformal symmetry. For the simplest case of the four point function for $[0, 2, 0]$ chiral primary operators this was undertaken in [19] and applied to determine the one-loop anomalous dimensions for all operators with lowest order twist two.

The procedure adopted in [19] is somewhat involved and does not simply generalise to correlation functions of more general chiral primary operators. As was shown in [19] the superconformal Ward identities are simplified if they are expressed in terms of new variables $x, \bar{x}$ rather than the usual conformal invariants. In terms of the standard correspondence for the space–time coordinates $x^a \to \mathrm{x} = x^a\sigma_a$, where x is a 2×2 spinorial matrix such that $\det \mathrm{x} = -x^2$, then, for four points x_1, x_2, x_3, x_4 and $x_{ij} = x_i - x_j$, $x, \bar{x}$ may be defined, as shown in [20], as the eigenvalues of $\mathrm{x}_{12}\mathrm{x}_{42}{}^{-1}\mathrm{x}_{43}\mathrm{x}_{13}{}^{-1}$. By conformal transformations we may choose a frame such that $\mathrm{x}_2 = 0$, $\mathrm{x}_3 = \infty$, $\mathrm{x}_4 = 1$ and $\mathrm{x}_1 = \left(\begin{smallmatrix} x & 0 \\ 0 & \bar{x} \end{smallmatrix}\right)$. The two conformal invariants are then given in terms of $x, \bar{x}$ by,[2]

$$u = \det\bigl(\mathrm{x}_{12}\mathrm{x}_{42}{}^{-1}\mathrm{x}_{43}\mathrm{x}_{13}{}^{-1}\bigr) = x\bar{x},$$

[2] Since $1 + u - v = x + \bar{x}$ and $1 + u^2 + v^2 - 2uv - 2u - 2v = (x - \bar{x})^2$ it is easy to invert these results to obtain $x, \bar{x}$ in terms of u, v up to the arbitrary sign of the square root $\sqrt{(x - \bar{x})^2}$. For any $f(u, v)$ there is a corresponding symmetric function $\hat{f}(x, \bar{x}) = \hat{f}(\bar{x}, x)$ such that $\hat{f}(x, \bar{x}) = f(u, v)$.