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Abelian sandpile model: A conformal field theory point of view

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Abstract

In this paper we derive the scaling fields in $c = -2$ conformal field theory associated with weakly allowed clusters in Abelian sandpile model and show a direct relation between the two models.

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Keywords: Conformal field theory; Self organized criticality; Sandpile model

1. Introduction

There exist some phenomena which naturally show power law behavior, that is, without fine tuning any parameter, the system shows behavior similar to the critical point, in contrast with the usual critical phenomena, where you should fine tune an external parameter like temperature to arrive at the critical point. These kind of phenomena are said to have self organized criticality [1]. Sandpile [1,2], surface growth [3] and river networks [4] are a few examples of such phenomena.

The concept of SOC was first introduced by Bak, Tang and Wiesenfeld [1]. While many other models have been found after that, still the Abelian sandpile model (ASM) is one of the simplest and most studied models. Despite its simplicity, it shows all the features of a

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self organized critical phenomenon, so huge amount of work has been done on this model [5–18]. Many analytic results have also been derived, for example the probability of different heights and many specific clusters are calculated explicitly [6,17]. Also the relation of this model to other known models has been discussed. First of all, there is a connection between ASM and spanning trees [7]. Then other models such as dense polymers, Scheidegger's model of river networks and $q \to 0$ limit of q-state Potts models are related to ASM [19].

These statistical models show conformal symmetry while they are at their critical point. So it is natural to look for a conformal field theory (CFT) which corresponds to these systems. The CFT associated with these models is suggested to be $c = -2$, which belongs to a specific group of CFT's, known as logarithmic conformal field theories (LCFT's). In LCFT's, where correlation functions may have logarithmic terms in contrast with the ordinary CFT's, there exist pairs of fields with the same conformal weight, which mix under conformal transformations [20,21].

Maheiu and Ruelle [10] have found a way to relate ASM to $c = -2$ theory. They have found some operators in the LCFT model which correspond to different clusters in ASM. But the correspondence is shown only through correlation function and it is not clear why one should take this operator. This method was generalized by [14] to all clusters known as weakly allowed ones, though again only correlation functions were considered to derive the fields. In this paper we address this question and find a direct way to derive the operators from the action of $c = -2$, and hence connect ASM to $c = -2$ directly. The paper is organized as follows: in Section 2 we will briefly introduce the model and some analytical results including the result obtained in [10]. In Section 3 we derive the previously mentioned operators directly from the action of $c = -2$ model.

2. Abelian sandpile model

The Abelian sandpile model is defined on a square lattice of the size $L \times M$. To each site i, a height variable h_i is assigned. Height of sand at each site can take one of the values from the set $\{1, 2, 3, 4\}$. So the total number of different allowed configurations is equal to 4^{LM}. The dynamic of the system is defined as follows: at each step a random site i is selected and a grain of sand is added to that site. If the new height of sand becomes more than four, the column of sand is called unstable and topples, that is, four grains will leave the site and each of them will be added to one of the neighbors. So the total number of sands is conserved during the toppling process except at the boundaries where one or more grains leave the system.

The toppling process can be stated in perhaps more appropriate way. If the site i becomes unstable, h_j will be decreased by amount of Δ_{ij}, that is $h_j \to h_j - \Delta_{ij}$ where

$$\Delta_{ij} = \begin{cases} 4, & i = j; \\ -1, & i, j \text{ are neighbors;} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The matrix Δ_{ij} is called toppling matrix.

After a while the system reaches a steady state, in which it shows SOC. It has been shown that in this state, the number of different configurations the system accepts is $\det \Delta$