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Inequivalent quantizations of the three-particle Calogero model constructed by separation of variables

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Abstract

We quantize the 1-dimensional 3-body problem with harmonic and inverse square pair potential by separating the Schrödinger equation following the classic work of Calogero, but allowing all possible self-adjoint boundary conditions for the angular and radial Hamiltonians. The inverse square coupling constant is taken to be $g = 2\nu(\nu - 1)$ with $\frac{1}{2} < \nu < \frac{3}{2}$ and then the angular Hamiltonian is shown to admit a 2-parameter family of inequivalent quantizations compatible with the dihedral D_6 symmetry of its potential term $9\nu(\nu - 1)/\sin^2 3\phi$. These are parametrized by a matrix $U \in U(2)$ satisfying $\sigma_1 U \sigma_1 = U$, and in all cases we describe the qualitative features of the angular eigenvalues and classify the eigenstates under the D_6 symmetry and its S_3 subgroup generated by the particle exchanges. The angular eigenvalue λ enters the radial Hamiltonian through the potential $(\lambda - \frac{1}{4})/r^2$ allowing a 1-parameter family of self-adjoint boundary conditions at $r = 0$ if $\lambda < 1$. For $0 < \lambda < 1$ our analysis of the radial Schrödinger equation is consistent with previous results on the possible energy spectra, while for $\lambda < 0$ it shows that the energy is not bounded from below rejecting those U's admitting such eigenvalues as physically impermissible. The permissible self-adjoint angular Hamiltonians include, for example, the cases $U = \pm\mathbf{1}_2, \pm\sigma_1$, which are explicitly solvable and are presented in detail. The choice $U = -\mathbf{1}_2$ reproduces Calogero's quantization, while for the choice $U = \sigma_1$ the system is smoothly connected to the harmonic oscillator in the limit $\nu \to 1$.

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1. Introduction

The Calogero model [1,2] of N identical particles on the line subject to combined inverse square and harmonic interaction potential is extremely popular because of its exact solvability and its connections to many interesting problems in physics and mathematics. (See, for instance, [3] and references therein.) The Hamiltonian of the system is formally given by

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{i=2}^{N}\sum_{j=1}^{i-1}\left\{\frac{1}{4}m\omega^2(x_i - x_j)^2 + g(x_i - x_j)^{-2}\right\}. \tag{1.1}$$

After separation of the centre of mass, the $N = 2$ case reduces to the study of the 1-dimensional Schrödinger operator

$$H_y = -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\omega^2 y^2 + \frac{g}{2}y^{-2}. \tag{1.2}$$

As is widely known [4], the spectrum of H_y cannot be bounded from below if $g < -\frac{\hbar^2}{4m}$, and for this reason[2] Calogero assumed in his work that $g > -\frac{\hbar^2}{4m}$. For selecting the 'admissible wave functions' he imposed the criterion that the associated probability current should vanish at the locations where any two particles collide. This is intuitively reasonable if the inverse square potential is repulsive.

Mathematically speaking, the selection of admissible wave functions is equivalent to choosing a domain on which the Hamiltonian is self-adjoint. Concerning the 1-dimensional Hamiltonian H_y, it is known (see, e.g., [5,6] or the books [7,8]) that the choice of its self-adjoint domain is essentially unique if $g \geqslant \frac{3\hbar^2}{4m}$, but there exits a family of different possibilities parametrized by a 2×2 unitary matrix if $g < \frac{3\hbar^2}{4m}$. In the corresponding quantizations of the $N = 2$ Calogero model the probability current does not in general vanish at the coincidence of the coordinates of the particles. Heuristically speaking, if $0 < g < \frac{3\hbar^2}{4m}$, then the particles can pass through each other by a tunneling effect despite the infinitely high repulsive potential barrier. Since this phenomenon refers to the interaction of any pairs of particles, one may expect it to occur also in the N particle Calogero model.

The purpose of the present paper is to explore the inequivalent quantizations of the Calogero model under the assumption

$$-\frac{\hbar^2}{4m} < g < \frac{3\hbar^2}{4m} \quad (g \neq 0) \tag{1.3}$$

focusing on the simplest non-trivial case of three particles. We shall use separation of variables to define the quantizations. To explain the main point, recall that the Hamiltonian

[2] The energy can still be bounded from below if $g = -\frac{\hbar^2}{4m}$, but this case would require a separate treatment.