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‘Square root’ of the Maxwell Lagrangian versus confinement in general relativity

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ABSTRACT

We employ the ‘square root’ of the Maxwell Lagrangian (i.e. $\sqrt{F_{\mu\nu}F^{\mu\nu}}$), coupled with gravity to search for the possible linear potentials which are believed to play role in confinement. It is found that in the presence of magnetic charge no confining potential exists in such a model. Confining field solutions are found for radial geodesics in pure electrically charged Nariai–Bertotti–Robinson (NBR)-type spacetime with constant scalar curvature. Recently, Guendelman, Kaganovich, Nissimov and Pacheva (2011) [7] have shown that superposed square root with standard Maxwell Lagrangian yields confining potentials in spherically symmetric spacetimes with new generalized Reissner–Nordström–de Sitter/anti-de Sitter black hole solutions. In NBR spacetimes we show that confining potentials exist even when the standard Maxwell Lagrangian is relaxed.

A power-law extension of the Maxwell action coupled with gravity was considered by [1–3]

$$I=\frac{1}{2}\int dx^4\sqrt{-g}\big(R-2\Lambda-\alpha\mathcal{F}^s\big), \tag{1}$$

in which s and α are real constants, $\mathcal{F}=F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and Λ stands for the cosmological constant. The first study with this form of nonlinear electrodynamic (NED) was made in spherical symmetry and ever since many authors have considered different aspects/applications of this action [2]. Although the original paper [3] considered a conformally invariant action (i.e. $s=d/4$) this requirement was subsequently relaxed. It was shown that $s=1/2$ raised problems in connection with the energy conditions [4] and for this reason it was abandoned. Nielsen and Olesen [5] proposed such a magnetic ‘square root’ Lagrangian (i.e. $\sqrt{F_{\mu\nu}F^{\mu\nu}}$) in string theory while ’t Hooft [6] highlighted a linear potential term to be effective toward confinement. More recently Guendelman et al. [7] investigated confining electric potentials in black hole spacetimes in the presence of the standard Maxwell Lagrangian.

In this Letter we suppress the standard Maxwell Lagrangian, keeping only the ‘square root’ of the Maxwell Lagrangian, to search for confining potentials. It is known that under the scale transformation, i.e. $x_\mu\to\lambda x_\mu$, $A_\mu\to\frac{1}{\lambda}A_\mu$ ($\lambda=$ const.) in $d=4$ the latter doesn’t remain invariant. Even in this reduced form we prove the existence of such potentials in some spacetimes identified as the Nariai–Bertotti–Robinson (NBR)-type spacetime. Due to the absence of Maxwell Lagrangian $\sim F_{\mu\nu}F^{\mu\nu}$, however, the Coulomb potential will be missing in our formalism. We choose the case $s=1/2$ in $d=4$ with a general line element

$$ds^2=-f(r)\,dt^2+\frac{dr^2}{f(r)N(r)^2}+R(r)^2\big(d\theta^2+\sin^2\theta\,d\varphi^2\big), \tag{2}$$

where $f(r)$, $N(r)$ and $R(r)$ are three unknown functions of r. Our choice of Maxwell 2-form is

$$F=E(r)\,dt\wedge dr+P\sin(\theta)\,d\theta\wedge d\varphi \tag{3}$$

in which P stands for the magnetic charge constant and $E(r)$ is to be determined. From the variational principle the nonlinear Maxwell equation reads

$$d\left(\frac{{}^\star\mathbf{F}}{\sqrt{\mathcal{F}}}\right)=0, \tag{4}$$

in which ${}^\star\mathbf{F}$ is dual of $\mathbf{F}$. Using the line element one finds

$${}^\star\mathbf{F}=ENR^2\sin\theta\,d\theta\wedge d\varphi-\frac{P}{NR^2}\,dt\wedge dr, \tag{5}$$

and

$$\mathcal{F}=-2E^2N^2+\frac{2P^2}{R^4}. \tag{6}$$

The nonlinear Maxwell equation yields

$$\frac{ENR^2}{\sqrt{-2E^2N^2+\frac{2P^2}{R^4}}}=\frac{\beta}{\sqrt{2}} \tag{7}$$

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where β is an integration constant. This equation admits a solution for the electric field as

$$E=\frac{P\beta}{NR^2\sqrt{R^4+\beta^2}}, \tag{8}$$

and therefore

$$\mathcal{F}=\frac{2P^2}{R^4+\beta^2}. \tag{9}$$

We note here that $\mathcal{F}$ is positive which is needed for our choice of square root expression. Variation of the action with respect to $g_{\mu\nu}$ gives Einstein–Maxwell equations

$$G_\mu^\nu+\Lambda g_\mu^\nu=T_\mu^\nu \tag{10}$$

in which

$$T_\nu^\mu=-\frac{\alpha}{2}\left(\delta_\nu^\mu\sqrt{\mathcal{F}}-\frac{2(F_{\nu\lambda}F^{\mu\lambda})}{\sqrt{\mathcal{F}}}\right). \tag{11}$$

Explicitly we find

$$T_t^t=T_r^r=-\frac{\alpha}{\sqrt{2}}\left(\frac{P\sqrt{R^4+\beta^2}}{R^4}\right), \tag{12}$$

and

$$T_\theta^\theta=T_\phi^\phi=\frac{\alpha}{\sqrt{2}}\frac{P\beta^2}{R^4\sqrt{R^4+\beta^2}}. \tag{13}$$

Having $T_t^t=T_r^r$ means that $G_t^t=G_r^r$ which leads to $N(r)=C$ and $R(r)=r$. Note that C is an integration constant which is set for convenience to $C=1$. The Einstein equations admit a black hole solution for the metric function given by

$$f(r)=1-\frac{2m}{r}-\frac{\Lambda}{3}r^2-\frac{P\alpha}{\sqrt{2}r}\int\sqrt{1+\frac{\beta^2}{r^4}}\,dr. \tag{14}$$

Here by using the expansion $\sqrt{1+t}=1+\frac{1}{2}t-\frac{1}{8}t^2+\mathcal{O}(t^3)$ for $|t|<1$ one finds for large r (i.e. $\frac{r^4}{\beta^2}>1$)

$$f(r_{large})=1-\frac{P\alpha}{\sqrt{2}}-\frac{2m}{r}-\frac{\Lambda}{3}r^2+\frac{P\alpha\beta^2\sqrt{2}}{12r^4}+\mathcal{O}\left(\frac{1}{r^8}\right), \tag{15}$$

and for small r (i.e. $\frac{r^4}{\beta^2}<1$) we rewrite $\int\sqrt{1+\frac{\beta^2}{r^4}}\,dr=\int\frac{\beta}{r^2}\times\sqrt{1+\frac{r^4}{\beta^2}}\,dr$ which implies

$$f(r_{small})=1-\frac{2m}{r}+\frac{P\alpha\beta}{\sqrt{2}r^2}-\left(\frac{\Lambda}{3}+\frac{P\alpha\sqrt{2}}{12\beta}\right)r^2+\frac{P\alpha\sqrt{2}}{112\beta^3}r^6+\mathcal{O}\left(r^{10}\right), \tag{16}$$

where m is an integration constant related to mass. The Ricci scalar of the spacetime is given by

$$R=2+\frac{4m}{r^3}-\frac{2\sqrt{2}\alpha P\sqrt{r^4+\beta^2}}{r^4}+\frac{\sqrt{2}P\alpha}{\sqrt{r^4+\beta^2}}-\frac{\sqrt{2}P\alpha}{r^3}\int\sqrt{1+\frac{\beta^2}{r^4}}\,dr, \tag{17}$$

which at infinity is convergent while at $r=0$ is singular. For a moment in order to see the structure of the electromagnetic field (3) we resort to the flat spacetime given by the line element

$$ds^2=-dt^2+dr^2+r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right). \tag{18}$$

The electric field reads as

$$E=\frac{P}{r^2\sqrt{1+\frac{r^4}{\beta^2}}} \tag{19}$$

which results in the potential

$$V=-P\int\frac{dr}{r^2\sqrt{1+\frac{r^4}{\beta^2}}}. \tag{20}$$

Here also we use the expansion $\frac{1}{\sqrt{1+t}}=1-\frac{1}{2}t+\frac{3}{8}t^2+\mathcal{O}(t^3)$ for $|t|<1$ to obtain

$$V(r_{small})=-P\int\frac{dr}{r^2}\left(1+\frac{r^4}{\beta^2}\right)^{-\frac{1}{2}}=\frac{P}{r}+\frac{Pr^3}{6\beta^2}-\frac{3Pr^7}{56\beta^4}+\mathcal{O}\left(r^{11}\right), \tag{21}$$

for small r and

$$V(r_{large})=-P\beta\int\frac{dr}{r^4}\left(1+\frac{\beta^2}{r^4}\right)^{-\frac{1}{2}}=\frac{P\beta}{3r^3}-\frac{P\beta^3}{14r^7}+\frac{3P\beta^5}{88r^{11}}+\mathcal{O}\left(\frac{1}{r^{15}}\right), \tag{22}$$

for large r. It is readily seen that the magnetic charge P is indispensable for an electric solution to exist in the flat spacetime.

Now, going back to the curved space metric ansatz

$$ds^2=-f(r)\,dt^2+\frac{dr^2}{f(r)}+r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right) \tag{23}$$

one obtains, for $\beta=0$, the exact solution from (14) as

$$f(r)=1-\frac{2m}{r}-\frac{\Lambda}{3}r^2-\frac{P\alpha}{\sqrt{2}}, \tag{24}$$

with $E(r)=0$. Such a metric represents a global monopole [8] with a deficit angle which is valid only for $P\neq 0$. We note that this represents a non-asymptotically flat black hole with mass, cosmological constant and global monopole charge.

For the case of pure electric field let us consider now in (3) $P=0$ and due to the sign problem we revise our square root term as $\sqrt{-F_{\mu\nu}F^{\mu\nu}}$ in the action. Further, to remove the ambiguity in arbitrariness of $E(r)$ from the Maxwell equation (4) we require that the spacetime has constant scalar curvature. This restricts our $E(r)$ only to be a constant. This yields with reference to the metric ansatz (2), as a result of the Maxwell equation, for the choice $N(r)=1$ that one obtains $R(r)=r_0=\text{constant}$ and $E(r)=E_0=\text{constant}$.

The tt and rr components of the Einstein equations yield

$$\Lambda=\frac{1}{r_0^2}, \tag{25}$$

so that the solution for $f(r)$ takes the form

$$f=-\left(\Lambda+\frac{\alpha E_0}{\sqrt{2}}\right)r^2+C_1r+C_2 \tag{26}$$

where C_1 and C_2 are constants of integration. With this $f(r)$ the line element reads

$$ds^2=-f(r)\,dt^2+\frac{dr^2}{f(r)}+r_0^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right) \tag{27}$$

in which the electric field (E_0) and cosmological constant ($\Lambda=\frac{1}{r_0^2}$) are both essential parameters. By setting $E_0=0=C_1$, it reduces