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A resummable β-function for massless QED

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Abstract

Within the set of schemes defined by generalized, manifestly gauge invariant exact renormalization groups for QED, it is argued that the β-function in the four-dimensional massless theory cannot possess any nonperturbative power corrections. Consequently, the perturbative expression for the β-function must be resummable. This argument cannot be extended to flows of the other couplings or to the anomalous dimension of the fermions and so perturbation theory does not define a unique trajectory in the critical surface of the Gaussian fixed point. Thus, resummability of the β-function is not inconsistent with the expectation that a non-trivial fixed point does not exist.

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The resummability,[1] or otherwise, of the perturbative series for the β-functions and anomalous dimension(s) in some quantum field theory (QFT) is intimately related to the nonperturbative question of renormalizability. This is beautifully explained in [1] (see also [2]), and we here recall the main points. The formalism best suited to understanding such issues is the Exact Renormalization Group (ERG), which is essentially the continuous version of Wilson's RG [3,4]. A fundamental ingredient of this approach is the implementation of a momentum cutoff, such that all modes above the cutoff scale are regularized. For the following discussion, we consider two cutoff scales. First, there is the bare scale, Λ_0, which provides an overall cutoff to the theory. As we shall see, for nonperturbatively renormalizable theories, this scale is an artificial construction, and it is misleading to identify the action at this scale as a boundary condition that can be chosen, arbitrarily. (The same is not true for nonrenormalizable theories.) We now integrate out degrees of freedom between the bare scale and a lower, 'effective' scale, Λ. As we perform this procedure, the bare action evolves into the Wilsonian effective action, S_Λ, in such a way that the partition function stays the same. The Wilsonian effective action can be thought of as parametrizing the interactions relevant to the effective scale. The ERG equation states how the Wilsonian effective action changes with the effective scale.

One of the most important uses of the ERG equation is to find QFTs which are nonperturbatively renormalizable, in other words theories for which Λ_0 can be send to infinity (this is called taking the continuum limit). Scale independent renormalizable theories follow immediately from fixed points of the ERG equation. To see this, suppose that we rescale all dimensionful quantities to dimensionless ones, by dividing by Λ raised to the appropriate scaling dimension. Now, fixed points of the ERG can be immediately identified with renormalizable theories: as a consequence of our rescalings, independence of Λ implies independence of all scales; independence of all scales trivially implies independence of Λ_0, and so obviously we can send Λ_0 to infinity!

Scale dependent renormalizable theories can be constructed by considering flows out of any of the fixed points, along the associated relevant/marginally relevant directions. The Wilsonian effective actions lying on these 'Renormalized Trajectories' (RTs) [3] are self-similar, meaning that all dependence on Λ appears only through the renormalized relevant/marginally rel-

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[1] Throughout this Letter we have in mind Borel resummability, though our conclusions should not depend on this choice.

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evant couplings and anomalous dimension(s).[2] Self-similarity implies renormalizability, since there is no explicit dependence on Λ/Λ_0. Note that, along an RT, the theory is completely specified by the choice of fixed point, and the integration constants or 'rates' associated with the relevant/marginally relevant directions. In the limit $\Lambda \to \infty$, the theory sinks back into the appropriate fixed point. Thus, if we wish to consider the action at some arbitrarily high 'bare' scale, we must *compute* it using the flow equation, given our aforementioned choices. Indeed, the 'bare action' in this context is the perfect action [5] in the vicinity of the ultraviolet (UV) fixed point. This is in contrast to a nonrenormalizable trajectory, where we can simply chose some bare action, and use it as the boundary condition for the flow.

One of the benefits of viewing renormalization in this way is that, along RTs, we can compute directly in terms of renormalized quantities, without any mention of the bare scale or the bare action. To do this, we employ renormalization conditions for the relevant and marginally relevant couplings and the anomalous dimension(s) directly at the effective scale, Λ. So, if a non-trivial RT were to exist in QED (we are not claiming that one does in four dimensions, where the gauge coupling is marginally *irrelevant*, but one does in three dimensions) then we would define the gauge coupling—which we denote by g and not e to avoid later confusion—simply by writing the gauge kinetic term as

$$\frac{1}{g^2(\Lambda)} \int d^D x\, F_{\mu\nu} F_{\mu\nu},$$

at all scales. Note that we have scaled the coupling out of the gauge field. In the manifestly gauge invariant approach that we will later adopt, this will have the pleasant effect of guaranteeing that the gauge field does not suffer from field strength renormalization [6]. Throughout this Letter we work in Euclidean space, so there is no distinction between upper and lower indices.

Let us now consider a massless theory, about which it is supposed that all we know is that its ERG trajectory lies in the critical surface of some fixed point. Since this trajectory is flowing *into* a fixed point, and we have not specified whether or not the trajectory happens to have emanated from some other fixed point in the UV, we do not know, a priori, whether the theory is renormalizable or not. To be concrete, we will suppose that this infrared fixed point is the Gaussian one, that this fixed point possesses a single marginally irrelevant coupling, g, and that there is a single anomalous dimension (just as there is in our manifestly gauge invariant approach to QED in four dimensions).

Let us now do perturbation theory in the vicinity of the Gaussian fixed point, within the critical surface. For reasons that will become apparent, we will attempt to write the action in self-similar form. Consequently, our renormalization conditions involve conditions for only the coupling, g, and the anomalous dimension, γ, specified at the scale Λ. We have assumed, temporarily, that no reference to the bare scale/bare action is necessary. Computing the full perturbative solution to the theory, we find that everything can be written in renormalized terms [6], defining an apparently unique, self-similar trajectory in the critical surface of the Gaussian fixed point. Were it really the case that this trajectory were both self-similar and unique, then this would suggest the existence of a UV fixed point, out of which an RT can be constructed that flows into the Gaussian fixed point. However, as emphasised in [1], this picture is generally false. In the specific case of scalar field theory in four dimensions, the perturbative series for the flows of the n-point couplings (the β-functions) of the theory are not resummable, and so do not unambiguously define functions. The reason for this is UV renormalons[3] (see [7] for a review of renormalons): perturbation theory by itself is not well defined, but must be supplemented by exponentially small terms which, in QED, take the form

$$\frac{\Lambda}{\Lambda_0} \sim \mathrm{e}^{-1/2\beta_1 g^2(\Lambda)} + \cdots, \tag{1}$$

where β_1 is the one-loop β-function and the ellipsis denotes higher order corrections ('the' β-function refers to the flow of g). The left-hand side of this expression makes it immediately clear that self-similarity of our trajectory is violated: there is explicit dependence on Λ_0. We can always write such power corrections in terms of g, as we have done on the right-hand side, but the prefactor will, of course, depend on Λ_0. Within perturbation theory, we can blithely take the limit $\Lambda_0 \to \infty$, but at some point have to face up to the fact that this procedure is not well defined, if we hope to draw reliable nonperturbative conclusions.

In light of this discussion, the main result of this Letter is rather unexpected: it will be demonstrated, in massless QED in four dimensions that, given a particular definition of the coupling, the perturbative series for the β-function *cannot be supplemented by terms of type* (1) *and its generalizations*. Since our ERG equation is perfectly well defined, and since we can choose a perfectly well defined boundary condition (bare action) we deduce that the perturbative β-function must be resummable. Nevertheless, our earlier conclusions are still intact, since our argument does not apply to the other couplings of the theory or to the anomalous dimension of the fermions. Consequently, perturbation theory still does not specify a unique, self-similar trajectory within the critical surface of the Gaussian fixed point, and so there is no suggestion that a UV fixed point exists.

Note, though, that matters could be much more interesting in the Wess–Zumino model. First, we note that all couplings belonging to the superpotential are protected from flowing by the nonrenormalization theorem. Secondly, it seems as though we can apply the arguments of this Letter to show that the perturbative series for the anomalous dimension is resummable. (By scaling the field strength renormalization out of the two-point

[2] Any masses are included in our definition of couplings.

[3] Throughout this Letter, we will loosely refer to a renormalon as *any* singularity of the Borel transform, rather than using the strict definition [7], which defines renormalons as those singularities related to large or small loop momentum behaviour.