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Quantum criticality and Yang–Mills gauge theory

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ABSTRACT

We present a family of nonrelativistic Yang–Mills gauge theories in $D+1$ dimensions whose free-field limit exhibits quantum critical behavior with gapless excitations and dynamical critical exponent $z=2$. The ground state wavefunction is intimately related to the partition function of relativistic Yang–Mills in D dimensions. The gauge couplings exhibit logarithmic scaling and asymptotic freedom in the upper critical spacetime dimension, equal to $4+1$. The theories can be deformed in the infrared by a relevant operator that restores Poincaré invariance as an accidental symmetry. In the large-N limit, our nonrelativistic gauge theories can be expected to have weakly curved gravity duals.

We present a class of nonrelativistic Yang–Mills gauge theories which exhibit anisotropic scaling between space and time. Our motivation originates from several different areas of physics, which have been experiencing a stimulating confluence of theoretical ideas recently: condensed matter theory, in particular quantum critical phenomena, string theory, and gauge–gravity duality.

In our study, we consider the case of $D+1$ spacetime dimensions, continuing the viewpoint advocated in [1] that the interface of condensed matter and string theory is best studied from the vantage point of arbitrary number of dimensions, even though practical applications to condensed matter are likely to be expected only for $D \leqslant 3$.

The theories presented here are candidates for the description of new universality classes of quantum critical phenomena in various dimensions. They combine the idea of non-Abelian gauge symmetry, mostly popular in relativistic high-energy physics, with the concept of scaling with nonrelativistic values of the dynamical critical exponent, $z \neq 1$.[1] In combination, this anisotropic scaling together with Yang–Mills symmetry opens up a new perspective on gauge theories, changing some of the basic features of relativistic Yang–Mills such as the critical dimension in which the theory exhibits logarithmic scaling.

Since our theories can be constructed for any compact gauge group, the choice of the $SU(N)$, $SO(N)$ or $Sp(N)$ series yields a class of theories with anisotropic scaling and a natural large-N expansion parameter. These theories can then be expected to have weakly-curved gravitational duals, perhaps leading to new realizations of the AdS/nonrelativistic CFT correspondence which has attracted considerable attention recently [3–6]. Finally, it turns out that our theories are intimately related to relativistic theories in one fewer dimension, and therefore can shed some new light on the dynamics of the relativistic models.

1. Theories of the Lifshitz type

We work on a spacetime of the form $\mathbf{R}\times\mathbf{R}^D$, with coordinates t and $\mathbf{x}\equiv(x^i)$, $i=1,\dots D$, equipped with the flat spatial metric δ_{ij} (and the metric $g_{tt}=1$ on the time dimension). The theories proposed here are of the Lifshitz type, and exhibit fixed points with anisotropic scaling characterized by dynamical critical exponent z (see, e.g., [7]),

$$\mathbf{x}\to b\mathbf{x}, \qquad t\to b^z t. \tag{1.1}$$

We will measure dimensions of operators in the units of spatial momenta, defining

$$[x^i]=-1, \qquad [t]=-z. \tag{1.2}$$

The prototype of a quantum field theory with nontrivial dynamical exponent z is the theory of a single Lifshitz scalar $\phi(\mathbf{x},t)$. In its simplest incarnation, this theory is described by the following action,

$$S=\frac{1}{2}\int dt\,d^D\mathbf{x}\left\{(\dot\phi)^2-\frac{1}{4\kappa^2}(\Delta\phi)^2\right\}, \tag{1.3}$$

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[1] A different proposal for a quantum critical gauge theory, in $2+1$ dimensions and with $\mathcal{G}=SU(2)$, was made in [2].

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where $\Delta \equiv \partial_i\partial_i$ is the spatial Laplacian. Throughout most of the Letter we adhere to the nonrelativistic notation, and denote the time derivative by "$\dot{}$".

The Lifshitz scalar is a free-field fixed point with $z = 2$. The engineering dimension of ϕ is $[\phi] = D/2 - 1$, i.e., the same as the dimension of the relativistic scalar in D spacetime dimensions, implying an interesting shift in the critical dimensions of the $z = 2$ system compared to its relativistic cousin.

Note that the potential term in the Lifshitz action (1.3) is of the form

$$\left(\frac{\delta W[\phi]}{\delta\phi}\right)^2, \tag{1.4}$$

where W is the Euclidean action of a massless relativistic scalar in D dimensions,

$$W[\phi] = \frac{1}{2\kappa}\int d^D\mathbf{x}\,(\partial_i\phi\partial_i\phi). \tag{1.5}$$

When Wick rotated to imaginary time $\tau = it$, the action can be written as a perfect square,

$$S = \frac{i}{2}\int d\tau\, d^D\mathbf{x}\left\{\left(\partial_\tau\phi + \frac{1}{2\kappa}\Delta\phi\right)^2\right\}, \tag{1.6}$$

because the cross-term in (1.6) is a total derivative, $\dot\phi\Delta\phi/\kappa = -\dot W$, and can be dropped.

The coupling $\kappa \in [0,\infty)$ parametrizes a line of fixed points. If we wish, we can absorb κ into the rescaling of the time coordinate and a rescaling of ϕ.

In the original condensed-matter applications [8–10], the anisotropy is between different spatial dimensions, and the Lifshitz scalar is designed to describe the tricritical point at the juncture of the phases with a zero, homogeneous and spatially modulated condensate.

2. Yang–Mills theory and quantum criticality

Our nonrelativistic gauge theory in $D + 1$ dimensions will be similarly associated with relativistic Yang–Mills in D dimensions.

Our gauge field is a one-form on spacetime, with spatial components $A_i = A_i^a(x^j,t)T_a$ and a time component $A_0 = A_0^a(x^i,t)T_a$. The Lie algebra generators T_a of the gauge group $\mathcal{G}$ (which we take to be compact and simple or a $U(1)$) satisfy commutation relations $[T_a, T_b] = i f_{ab}{}^c T_c$. We normalize the trace on the Lie algebra of $\mathcal{G}$ by $\mathrm{Tr}(T_aT_b) = \frac{1}{2}\delta_{ab}$.

Our theory will be invariant under gauge symmetries

$$\delta_\epsilon A_i = \left(\partial_i\epsilon^a + f_{bc}{}^a A_i^b\epsilon^c\right)T_a \equiv D_i\epsilon,$$
$$\delta_\epsilon A_0 = \dot\epsilon - i[A_0,\epsilon]. \tag{2.1}$$

Gauge-invariant Lagrangians will be constructed from the field strengths

$$E_i = \left(\dot A_i^a - \partial_i A_0^a + f_{bc}{}^a A_i^b A_0^c\right)T_a = \dot A_i - \partial_i A_0 - i[A_i, A_0],$$
$$F_{ij} = \left(\partial_i A_j^a - \partial_i A_i^a + f_{bc}{}^a A_i^b A_j^c\right)T_a$$
$$= \partial_i A_j - \partial_i A_i - i[A_i, A_j]. \tag{2.2}$$

We will now construct a theory which has $z = 2$ in the free field limit. The engineering dimensions of the gauge field components at the corresponding Gaussian fixed point will be

$$[A_i] = 1, \qquad [A_0] = 2. \tag{2.3}$$

The Lagrangian should contain a kinetic term which is quadratic in first time derivatives, and gauge invariant. The unique candidate for this kinetic term is $\mathrm{Tr}(E_iE_i)$, of dimension $[\mathrm{Tr}(E_iE_i)] = 6$. One can then follow the strategy of effective field theory, and add all possible terms with dimensions $\leqslant 6$ to the Lagrangian. This would allow terms such as $\mathrm{Tr}(F_{ij}F_{k\ell}F_{\ell i})$, $\mathrm{Tr}(D_iF_{jk}D_iF_{jk})$, $\mathrm{Tr}(D_iF_{ik}D_jF_{jk})$ (all of dimension six), a term $\mathrm{Tr}(F_{ij}F_{ij})$ of dimension four, etc. One could indeed define the theory in this fashion, study the renormalization-group (RG) behavior in the space of all the couplings, and look for possible fixed points. This interesting problem is beyond the scope of the present Letter. Instead, we pursue a different strategy, and limit the number of independent couplings in a way compatible with renormalization. The trick that we will use is familiar from a variety of areas of physics, such as dynamical critical systems [11,12], stochastic quantization [13,14], and nonequilibrium statistical mechanics.

Inspired by the structure of the Lifshitz scalar theory, we take our action to be

$$S = \frac{1}{2}\int dt\, d^D\mathbf{x}\left\{\frac{1}{e^2}\mathrm{Tr}(E_iE_i) - \frac{1}{g^2}\mathrm{Tr}\big((D_iF_{ik})(D_jF_{jk})\big)\right\}. \tag{2.4}$$

This is a Lagrangian with $z = 2$ and no Galilean invariance. As a result, there is no symmetry relating the kinetic term and the potential term, and therefore no *a priori* relation between the renormalization of the two couplings e and g. The potential term is again the square of the equation of motion that follow from an action: the relativistic Yang–Mills in D Euclidean dimensions. When a theory in $D + 1$ dimensions is so constructed from the action of a theory in D dimensions, we will say that it *satisfies the detailed balance condition*, borrowing the terminology common in nonequilibrium dynamics.

3. At the free-field fixed point with $z = 2$

The free-field fixed point will be obtained from (2.4) by taking e and g simultaneously to zero. Keeping both the kinetic and the potential term finite in this limit requires rescaling the gauge field, $\tilde A_i^a \equiv A_i^a/\sqrt{eg}$, and keeping $\tilde A_i^a$ finite as we take e and g to zero. This gives

$$S = \frac{1}{2}\int dt\, d^D\mathbf{x}\left\{\frac{g}{e}\mathrm{Tr}(\tilde E_i\tilde E_i) - \frac{e}{g}\mathrm{Tr}\big((\partial_i\tilde F_{ik})(\partial_j\tilde F_{jk})\big)\right\}, \tag{3.1}$$

where $\tilde E_i$ and $\tilde F_{ij}$ are the linearized field strengths of $\tilde A_i$.

We see that there is actually a line of free fixed points, parametrized by the dimensionless ratio

$$\lambda = \frac{g}{e}. \tag{3.2}$$

As in the Lifshitz scalar theory, if we wish we can absorb λ into a rescaling of time, $t_{\mathrm{new}} = t/\lambda$.

The special properties of the Lifshitz scalar make it possible to determine the exact ground-state wavefunction [9],

$$\Psi\big[\phi(\mathbf{x})\big] = \exp\left\{-\frac{1}{4\kappa}\int d^D\mathbf{x}(\partial_i\phi\partial_i\phi)\right\}. \tag{3.3}$$

This Ψ is equal to $\exp(-W[\phi]/2)$, where $W[\phi]$ is the action (1.5) of the relativistic scalar in D dimensions. The norm $\int\mathcal{D}\phi(\mathbf{x})\Psi^*\Psi$ equals the partition function of this relativistic theory.

Similarly, we can relate the ground-state wavefunction of our $z = 2$ gauge theory to the partition function of relativistic Yang–Mills. The momenta and the Hamiltonian are

$$\tilde P_i^a = \frac{\lambda}{2}\tilde E_i^a,$$
$$H = \frac{2}{\lambda}\int dt\, d^D\mathbf{x}\,\mathrm{Tr}\left\{\tilde P_i\tilde P_i + \frac{1}{4}(\partial_i\tilde F_{ik})(\partial_j\tilde F_{jk})\right\}. \tag{3.4}$$