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Strings on a plane

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A B S T R A C T

Some features for a model of strings moving in $2+1$ spacetime dimensions are discussed that suggest applications to the quantum Hall effect. These features include edge-state configurations and anyons.

1. Mathematical model

Consider strings moving on a *flat* two-dimensional spatial surface (hence the target space is $D = 2 + 1$) with a one-dimensional spatial boundary. For constant $T\ \delta_{\mu\nu}$, constant $F\ \varepsilon_{\lambda\mu\nu}$, and induced world-sheet metric $g_{\alpha\beta}$, consider in particular the following model[1]:

$$\mathcal{L} = -T\delta_{\mu\nu}\sqrt{-\det g_{..}}\,g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{4}F\varepsilon_{\lambda\mu\nu}\varepsilon^{\alpha\beta}X^\lambda \partial_\alpha X^\mu \partial_\beta X^\nu. \tag{1}$$

We are interested in the evolution of initial string configurations on the plane, and the energy spectrum of the theory. Ultimately, we wish to consider interactions of the strings with the surface boundary, as well as interactions involving the splitting and joining of strings in the bulk. But for the most part these interactions and the details of the quantized model, along with the issue of whether (1) is an integrable system, will be considered in a later paper. Here we wish only to briefly consider the physics of the model as a framework for future applications.

The induced metric has the standard form

$$g_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X,$$
$$\det g_{..} = (\partial_\tau X \cdot \partial_\tau X)(\partial_\sigma X \cdot \partial_\sigma X) - (\partial_\tau X \cdot \partial_\sigma X)^2. \tag{2}$$

In addition to the well-known Nambu term, $\mathcal{L}_{\text{Nambu}} = \mathcal{L}|_{F=0}$, we have included a Chern–Simons term in the Lagrange density, $\mathcal{L}_{\text{CS}} = \mathcal{L}|_{T=0}$.[2] This CS term involves a two-form potential that corresponds, from an ambient spacetime perspective, to a constant field strength F on the spatial surface

$$A_{\mu\nu} = FX^\lambda \varepsilon_{\lambda\mu\nu}, \qquad \varepsilon^{\lambda\mu\nu}\partial_\lambda A_{\mu\nu} = 6F. \tag{3}$$

One motivation for introducing the CS term is that it may suggest and facilitate modifications for the three-string vertex that could alter the critical dimension for an interacting version of the model. But let us not look too far ahead.

The action for this model is world-sheet parameterization invariant, at least it is for the classical theory. It is also clearly invariant under Lorentz transformations on the bulk target manifold, bearing in mind that such transformations must also act on the target space boundary. In contrast, the world-sheet action is *not* manifestly target-space Poincaré invariant, since a constant shift $X^\lambda \to X^\lambda + C^\lambda$ adds to the action an amount

$$-\frac{1}{4}FC^\lambda\varepsilon_{\lambda\mu\nu}\int d\tau\, d\sigma\ \varepsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu = -\frac{1}{4}FC^\lambda\varepsilon_{\lambda\mu\nu}\int d\tau\, d\sigma\ \partial_\alpha\left(\varepsilon^{\alpha\beta}X^\mu \partial_\beta X^\nu\right).$$

This does not necessarily vanish for generic string configurations. But as indicated, it is a world-sheet boundary term, so the consequences of translational symmetry are not completely lost, as we shall see.

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[1] $\mu,\nu = 0,1,2$; $\delta_{\mu\nu} = \lceil 1,-1,-1 \rfloor$; $U\cdot V \equiv \delta_{\mu\nu}U^\mu V^\nu$; $\alpha,\beta = 0,1 \sim \tau,\sigma$; $\varepsilon^{01} = +1 = \varepsilon_{012}$.

[2] For example, see [2]. The same interaction appears in the 2-dimensional SO(3) pseudodual chiral σ-model of Zakharov and Mikhailov [15]. In that context, an infinite number of conservation laws were exhibited and discussed in [6]. Subsequently, such a term for string models has been mentioned *en passant* more than once in the literature [1,11].

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As usual, the classical equations of motion are

$$\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}}{\partial_\tau X^\mu}\right)+\frac{\partial}{\partial\sigma}\left(\frac{\partial\mathcal{L}}{\partial_\sigma X^\mu}\right)=\frac{\partial\mathcal{L}}{\partial X^\mu},\tag{4}$$

with the standard *form* of the boundary conditions for open strings,

$$0=\left.\frac{\partial\mathcal{L}}{\partial_\sigma X^\mu}\right|_{\text{any string end}}.\tag{5}$$

Although (4) is familiar in form, the right-hand side is absent for the Nambu string. But from the CS term in (1), we have

$$\frac{\partial\mathcal{L}}{\partial X^\mu}=-\frac{1}{2}F\varepsilon_{\mu\lambda\nu}\left(\partial_\tau X^\lambda\right)\left(\partial_\sigma X^\nu\right).\tag{6}$$

In addition, there are extra contributions to the left-hand side of (4) due to the CS term, namely,

$$E_\mu=F\varepsilon_{\mu\lambda\nu}\left(\partial_\tau X^\lambda\right)\left(\partial_\sigma X^\nu\right).\tag{7}$$

The net effect is to modify the usual Nambu equations of motion

$$\begin{aligned}&\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}_{\text{Nambu}}}{\partial_\tau X^\mu}\right)+\frac{\partial}{\partial\sigma}\left(\frac{\partial\mathcal{L}_{\text{Nambu}}}{\partial_\sigma X^\mu}\right)\\&\quad=\frac{\partial\mathcal{L}}{\partial X^\mu}-E_\mu\\&\quad=-\frac{3}{2}F\varepsilon_{\mu\lambda\nu}\left(\partial_\tau X^\lambda\right)\left(\partial_\sigma X^\nu\right).\end{aligned}\tag{8}$$

However, to put it differently, the right-hand side of the equation of motion (6) is in fact a total world-sheet divergence. So the equation of motion may still be written as a conservation law, similar to the Nambu string, but involving $\mathcal{L}_{\text{Nambu}}+\frac{3}{2}\mathcal{L}_{\text{CS}}\neq\mathcal{L}$

$$\frac{\partial}{\partial\tau}\left(\frac{\partial(\mathcal{L}_{\text{Nambu}}+\frac{3}{2}\mathcal{L}_{\text{CS}})}{\partial_\tau X^\mu}\right)+\frac{\partial}{\partial\sigma}\left(\frac{\partial(\mathcal{L}_{\text{Nambu}}+\frac{3}{2}\mathcal{L}_{\text{CS}})}{\partial_\sigma X^\mu}\right)=0.\tag{9}$$

This is of course tied to the world-sheet boundary contribution to the action produced by a constant translation of X. We will consider the physical implications of this in more detail shortly. In the meantime, we note the *explicit* form of the boundary conditions for open strings is also different than in the Nambu case. Here we have

$$\left.\frac{\partial\mathcal{L}_{\text{Nambu}}}{\partial_\sigma X^\mu}\right|_{\text{any string end}}=\left.\frac{1}{2}F\varepsilon_{\mu\lambda\nu}X^\lambda\partial_\tau X^\nu\right|_{\text{any string end}}\tag{10}$$

at both ends for each open string, if any. But note that this boundary condition does *not* require that the modified σ component, in the conservation law (9), vanish at the ends of a string. Rather, $\partial(\mathcal{L}_{\text{Nambu}}+\frac{3}{2}\mathcal{L}_{\text{CS}})/\partial_\sigma X^\mu|_{\text{any string end}}=-\frac{1}{4}F\varepsilon_{\mu\lambda\nu}X^\lambda\partial_\tau X^\nu|_{\text{any string end}}$.

As is well-known [8], it is possible to choose orthonormal world-sheet parameterizations. These are characterized by

$$(\partial_\tau X\cdot\partial_\tau X)+(\partial_\sigma X\cdot\partial_\sigma X)=0=\partial_\tau X\cdot\partial_\sigma X.\tag{11}$$

For such parameterizations the fully covariant equations of motion are

$$T\left(\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial\sigma^2}\right)X_\mu=\frac{3}{2}F\varepsilon_{\mu\lambda\nu}\partial_\tau X^\lambda\partial_\sigma X^\nu,\tag{12}$$

and the BCs become

$$0=\left[T\partial_\sigma X_\mu-\frac{1}{2}F\varepsilon_{\mu\lambda\nu}X^\lambda\partial_\tau X^\nu\right]_{\text{any string end}}.\tag{13}$$

So the nonlinearities of the equations are reduced to no worse than quadratic.

2. Spacetime Poincaré currents

Let us return to the issues raised by (9). The following combinations are conserved on the world-sheet using the covariant equations of motion

$$(\mathcal{P}_\mu)^\alpha=\left(\mathcal{P}_\mu^{\text{Nambu}}\right)^\alpha+\frac{3}{4}F\varepsilon^{\alpha\beta}\varepsilon_{\mu\lambda\kappa}X^\lambda\partial_\beta X^\kappa,\tag{14}$$

$$\begin{aligned}(\mathcal{M}_{\mu\nu})^\alpha&=X_\mu(\mathcal{P}_\nu)^\alpha-X_\nu(\mathcal{P}_\mu)^\alpha+\frac{3}{4}F\varepsilon^{\alpha\beta}\left(X\cdot X\varepsilon_{\mu\nu\lambda}\partial_\beta X^\lambda\right)\\&=\left(\mathcal{M}_{\mu\nu}^{\text{Nambu}}\right)^\alpha+\frac{3}{8}F\varepsilon^{\alpha\beta}\varepsilon_{\mu\nu\lambda}X^\lambda\partial_\beta(X\cdot X),\end{aligned}\tag{15}$$

with $\alpha=0,1\sim\tau,\sigma$, $(\mathcal{P}_\mu^{\text{Nambu}})^\alpha=\partial\mathcal{L}_{\text{Nambu}}/\partial_\alpha X^\mu$, and $(\mathcal{M}_{\mu\nu}^{\text{Nambu}})^\alpha=X_\mu(\mathcal{P}_\nu^{\text{Nambu}})^\alpha-X_\nu(\mathcal{P}_\mu^{\text{Nambu}})^\alpha$. We have already discussed the modification and conservation of the energy–momentum currents in the context of the equations of motion. Conservation of $\mathcal{P}_\mu$ requires the extra F-terms in (14). Similarly, the standard mechanical constructions for Lorentz currents,

$$\left(\mathcal{M}_{\text{mech}}^{\mu\nu}\right)^\alpha=X^\mu\left(\mathcal{P}^\nu\right)^\alpha-X^\nu\left(\mathcal{P}^\mu\right)^\alpha,\tag{16}$$

are seen *not* to be conserved upon using the equations of motion,

$$\begin{aligned}&\frac{\partial}{\partial\tau}\left(\mathcal{M}_{\text{mech}}^{\mu\nu}\right)^\tau+\frac{\partial}{\partial\sigma}\left(\mathcal{M}_{\text{mech}}^{\mu\nu}\right)^\sigma\\&\quad=\left(\frac{\partial}{\partial\tau}X^\mu\right)\left(\mathcal{P}^\nu\right)^\tau+\left(\frac{\partial}{\partial\sigma}X^\mu\right)\left(\mathcal{P}^\nu\right)^\sigma-(\mu\leftrightarrow\nu)\\&\quad=-\frac{3}{4}F\varepsilon_{\mu\nu\lambda}\left(\frac{\partial}{\partial\tau}(X\cdot X)\frac{\partial}{\partial\sigma}X^\lambda-(\tau\leftrightarrow\sigma)\right).\end{aligned}\tag{17}$$

Hence we have introduced the modifications in (15) to obtain conserved currents.

In an orthonormal parameterization of the world-sheet, these currents reduce to simple quadratic and cubic expressions in X

$$\left.\begin{matrix}(\mathcal{P}_\mu)^\alpha\\(\mathcal{M}_{\mu\nu})^\alpha\end{matrix}\right\}\underset{\text{orthonormal}}{\longrightarrow}\begin{cases}-T\partial^\alpha X_\mu+\frac{3}{4}F\varepsilon_{\mu\lambda\nu}\varepsilon^{\alpha\beta}X^\lambda\partial_\beta X^\nu,\\-T(X_\mu\partial^\alpha X_\nu-X_\nu\partial^\alpha X_\mu)\\\quad+\frac{3}{8}F\varepsilon_{\mu\nu\lambda}\varepsilon^{\alpha\beta}X^\lambda\partial_\beta(X\cdot X).\end{cases}\tag{18}$$

In this form, it is easiest to check conservation using (12).

These modifications of the various currents due to the F-terms suggest other ways to play with the structure of the currents. For example, the Lorentz currents (15) may be further altered, without loss of conservation, by adding two types of terms whose structure is peculiar to strings in $2+1$ spacetime dimensions, namely, $\varepsilon_{\mu\nu\lambda}\varepsilon^{\alpha\beta}\partial_\beta(X^\lambda f(X))$ and $\varepsilon_{\mu\nu\lambda}(\mathcal{P}^\lambda)^\alpha$. The first of these terms gives manifestly conserved topological world-sheet currents that do not alter the Lorentz charges, for closed strings. (Unless of course the topology of the spacetime is non-trivial, in which case winding configurations may appear. We will not consider this further here.) The second type of term is also conserved on the world-sheet, as a consequence of the covariant equations of motion, but in general it *would* change charges. Consider a one-parameter (u) family incorporating the second type of term

$$\left(\mathcal{M}_{\mu\nu}(u)\right)^\alpha=(\mathcal{M}_{\mu\nu})^\alpha+u\varepsilon_{\mu\nu\lambda}\left(\mathcal{P}^\lambda\right)^\alpha.\tag{19}$$

This construction is a bit strange for several reasons, so we will refer to it as "mock" spin. For one thing, the algebra of the modified charges is graded. That is to say, the algebra of the modified charges is an affine version of $so(2,1)$. For covariant parameterizations,

$$\begin{aligned}&\left[M_{\mu\nu}(u),M_{\kappa\lambda}(v)\right]\\&\quad=\left(\delta_{\mu\lambda}M_{\kappa\nu}(u+v)-\delta_{\nu\lambda}M_{\kappa\mu}(u+v)\right)-(\kappa\leftrightarrow\lambda)\\&\qquad+(\delta_{\mu\lambda}\delta_{\kappa\nu}-\delta_{\nu\lambda}\delta_{\kappa\mu})u\delta_{u+v}C,\end{aligned}\tag{20}$$